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FAKULTET TEHNIČKIH NAUKA

Clones of nondeterministic operations

-Doctoral dissertation-

Klonovi nedeterminističkih operacija

-Doktorska disertacija-

Mentor:
Prof. dr Jovanka Pantović

Kandidat:
Jelena Čolić Oravec

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Резиме на српском језику:	Ова теза представља преглед неких познатих и неколико нових резултата везаних за мреже тоталних, парцијалних, непотпуно специфицираних клонова и хиперклонова. Свакој парцијалној, непотпуно специфицираној и хипероперацији придружијемо одговарајућу тоталну операцију, и испитујемо тиме индукована потапања три мреже у одговарајуће мреже тоталних клонова. Потом познату Галоаову везу (Pol,Inv) између релација и операција модификујемо за парцијалне операције, НС операције и хипероперације и описујемо класе клонова непотпуно специфицираних и хипероперација које јако и слабо чувају дате релације. Такође наводимо неке познате резултате о мрежама на двоелементном скупу. Коначно, наводимо критеријуме комплетности за мреже тоталних и парцијалних клонова, а у случају хиперклонова и НС клонова описујемо четири класе коатома, одређених са четири класе Розенбергових релација.
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Sažetak

Motivacija

Ako neka funkcija ne daje rezultat za sve moguće ulazne vrednosti, kažemo da je parcijalno definisana. Takva funkcija f je preslikavanje iz skupa D u skup A, gde je A neprazan skup, a D pravi podskup od A^n. Ali šta ako se za $(a_1, \ldots, a_n) \in A^n \setminus D$ ne posmatra $f(a_1, \ldots, a_n)$ kao nedefinisana vrednost, već kao neodređena? Ova pretpostavka dovodi do značajne razlike kad je u pitanju kompozicija ovakvih operacija.

Neka je $A = \{0, 1\}$ i AND logička konjunkcija na A. Pretpostavimo da su f i g unarne operacije na A takve da je $f(0) = 0$, a vrednost $g(0)$ nije određena. Međutim, vrednost kompozicije $h(x) = \mathtt{AND}\big(f(x), g(x)\big)$ za $x = 0$ je određena i jednaka 0, odnosno važi

$$h(0) = \mathtt{AND}\big(f(0), g(0)\big) = \mathtt{AND}(0, g(0)) = 0.$$

Ovakav rezultat ima smisla s obzirom da, kao što je poznato, operacija AND uzima vrednost 0 kad god je bar jedan od argumenata jednak 0, bez obzira na vrednost drugog argumenta. Ipak, operacija h ne bi bila definisana za $x = 0$ ukoliko bismo smatrali da je $g(0)$ nedefinisana vrednost, tj. ako bismo date operacije i njihovu kompoziciju posmarali kao parcijalne operacije i standardnu kompoziciju parcijalnih operacija.

Kako možemo interpretirati ove neodređene izlazne vrednosti? Jedna mogućnost je da imamo više različitih izlaznih vrednosti za istu ulaznu vrednost, tj. da je izlazna vrednost neprazan podskup skupa A, što nam daje hiperoperacije. S druge strane, možemo izabrati neko $u \notin A$ i posmatrati ga kao proizvoljnu vrednost iz A, te na taj način dobijamo nepotpuno specificirane operacije.

Na dvoelementnom skupu ova dva koncepta su praktično ista, s obzirom da

važi $|A \cup \{u\}| = |\mathcal{P}(A) \setminus \{\emptyset\}| = 3$. Međutim, očigledno je da za $|A| > 2$ hiperoperacije imaju više mogućih izlaznih vrednosti nego nepotpuno specificirane operacije.

Hiperoperacije i nepotpuno specificirane operacije možemo koristiti za modeliranje nedeterminističkih procesa.

Na primer, u softverskim sistemima, ponovljeno izvršavanje istog programa može proizvesti različite rezultate, a takođe konkurentni procesi mogu imati različite redukcije. Ovakvo ponašanje se može predstaviti kao funkcija koja datom argumentu dodeljuje skup vrednosti. Takve funkcije su suštinski hiperoperacije. Svakako najpoznatiji primer u kome se ovakva funkcija pojavljuje jeste funkcija prelaska u definiciji nedeterminističkog konačnog automata.

S druge strane, u optimizaciji logičkih kola, ulazni podaci za koje izlazne vrednosti nisu specifikovane se nazivaju "don't care" uslovi i oni igraju važnu ulogu pri određivanju minimalnih disjunktivnih normalnih formi i dizajnu ekvivalentnih logičkih kola.

Klonovi

Ova disertacija predstavlja uporednu analizu mreža klonova totalnih operacija, parcijalnih operacija, nepotpuno specificiranih operacija i hiperoperacija.

Proučavanje klonova u univerzalnoj algebri je motivisano činjenicom da je skup termovskih operacija neke algebre $\mathbf{A} = (A, F)$ uvek klon operacija. Štaviše, skup C finitarnih operacija na A je klon ako i samo ako postoji algebra $\mathbf{A} = (A, F)$ takva da je C skup termovskih operacija te algebre. Međutim postoje i drugi načini da se definiše klon operacija.

Za bazni skup umesto proizvoljnog konačnog skupa A uzećemo skup $E_k = \{0, 1, \ldots, k-1\}$, $k \geq 2$. Tada je $f : E_k^n \to E_k$ n-arna (totalna) operacija na E_k. Označimo sa $O_k^{(n)}$ skup svih n-arnih operacija na E_k, a sa $O_k = \bigcup_{n \geq 1} O_k^{(n)}$ skup svih finitarnih operacija na E_k. Najpoznatija definicija klona je ona prema kojoj je $C \subseteq O_k$ klon ako sadrži sve projekcije, tj. operacije $e_i^n : E_k^n \to E_k$ date sa $e_i^n(x_1, \ldots, x_i, \ldots, x_n) = x_i$, i zatvoren je u odnosu na kompoziciju, tj. za $f \in O_k^{(n)}$ i $g_1, \ldots, g_n \in O_k^{(m)}$ važi $f(g_1, \ldots, g_n) \in C$, pri čemu za sve $\vec{x} \in E_k^m$

$$f(g_1, \ldots, g_n)(\vec{x}) = f\big(g_1(\vec{x}), \ldots, g_n(\vec{x})\big).$$

Kako je presek klonova ponovo klon, to za svaki skup operacija $F \subseteq O_k$ postoji najmanji klon koji ga sadrži. Označavamo ga sa $\langle F \rangle$ i kažemo da je to klon generisan sa F. Na ovaj način smo dobili algebarski operator zatvaranja, odakle sledi da skup svih klonova na E_k čini algebarsku mrežu $\mathcal{L}_k$ u odnosu na skupovnu inkluziju (čiji je najmanji element skup svih projekcija J_k, a najveći element skup svih operacija O_k). Otuda klonove možemo definisati i kao poduniverzume neke algebre. Kako bismo eksplicitno zadali takvu algebru, definišemo sledeće elementarne operacije na O_k, koje se još nazivaju i Mal'tsevljeve operacije. Neka su ζ, τ, Δ unarne i $*$ binarna operacija na O_k:

- za $f \in O_k^{(1)}$ neka je $\zeta f = \tau f = \Delta f = f$;
- za $f \in O_k^{(n)}, n \geq 2$, neka su $\zeta f, \tau f \in O_k^{(n)}$ i $\Delta f \in O_k^{(n-1)}$ date sa

$$(\zeta f)(x_1, x_2, \ldots, x_n) = f(x_2, \ldots, x_n, x_1)$$
$$(\tau f)(x_1, x_2, x_3, \ldots, x_n) = f(x_2, x_1, x_3, \ldots, x_n)$$
$$(\Delta f)(x_1, x_2, \ldots, x_{n-1}) = f(x_1, x_1, x_2, \ldots, x_{n-1});$$

- za $f \in O_k^{(n)}$ i $g \in O_k^{(m)}$ neka je $f * g \in O_k^{(m+n-1)}$ data sa

$$(f * g)(x_1, \ldots, x_{m+n-1}) = f\big(g(x_1, \ldots, x_m), x_{m+1}, \ldots, x_{m+n-1}\big).$$

Operacije ζ i τ nam daju sve permutacije promenljivih, Δ predstavlja izjednačavanje promenljivih, $*$ zamenu promenljive funkcijom. Algebra $\mathcal{O}_k = (O_k; *, \zeta, \tau, \Delta, e_1^2)$ se naziva puna algebra operacija. Sada je $C \subseteq O_k$ klon ako i samo ako je poduniverzum algebre $\mathcal{O}_k$.

Parcijalne operacije i nepotpuno specificirane operacije (NS operacije) ćemo posmatrati kao preslikavanja iz E_k^n u E_{k+1}, pri čemu ćemo kod parcijalnih operacija smatrati da je k nedefinisana vrednost, a kod NS operacija neodređena vrednost. Stoga ako zanemarimo razliku u interpretaciji izlazne vrednosti k, skupovi parcijalnih operacija (P_k) i NS operacija (I_k) su praktično isti. Razlika se uočava prilikom komponovanja ovih operacija. Kompoziciju parcijalnih operacija $f \in P_k^{(n)}$ i $g_1, \ldots, g_n \in P_k^{(m)}$, za sve $\vec{x} \in E_k^m$, definišemo sa

$$f(g_1, \ldots, g_n)(\vec{x}) = \begin{cases} f(g_1(\vec{x}), \ldots, g_n(\vec{x})), & \text{ako } \ g_i(\vec{x}) \in E_k, 1 \leq i \leq n \\ k, & \text{inače} \end{cases} .$$

To znači da kad god bar jedna od parcijalnih operacija $f, g_1, \ldots, g_n$ nije definisana, tada i njihova kompozicija nije definisana.

S druge strane, kompoziciju NS operacija $f \in I_k^{(n)}$ i $g_1, \ldots, g_n \in I_k^{(m)}$, za sve $\vec{x} \in E_k^m$, definišemo na sledeći način

$$f(g_1, \ldots, g_n)(\vec{x}) = \prod_{i=1}^{n} \big\{ f(\vec{y}) : \vec{y} \in E_k^n, y_i \sqsubseteq g_i(\vec{x}) \big\},$$

pri čemu je

$$\prod_{i=1}^{n} x_i = \begin{cases} x_1, & \text{ako } \ x_1 = x_2 = \ldots = x_n \\ k, & \text{inače} \end{cases}$$

i

$$\sqsubseteq = \big\{ (x, x) : x \in E_{k+1} \big\} \cup \big\{ (x, k) : x \in E_k \big\}.$$

Zato će u određenim slučajevima vrednost kompozicije biti iz E_k iako je za neke od NS operacija $f, g_1, \ldots, g_n$ izlazna vrednost k.

Hiperoperacije su preslikavanja koja svakoj n-torci elemenata iz E_k pridružuju neprazan podskup od E_k. Radi kraćeg zapisa skup nepraznih podskupova od E_k, $\mathcal{P}(E_k) \setminus \{\emptyset\}$, ćemo označavati sa $P_{E_k}^*$. Skup svih hiperoperacija na E_k označavamo sa H_k. Kompozicija hiperoperacija $f \in H_k^{(n)}$ i $g_1, \ldots, g_n \in H_k^{(m)}$, za sve $\vec{x} \in E_k^m$, se definiše kao

$$f(g_1, \ldots, g_n)(\vec{x}) = \bigcup_{i=1}^{n} \big\{ f(\vec{y}) : \vec{y} \in E_k^n, y_i \in g_i(\vec{x}) \big\}.$$

Slično kao u slučaju klonova totalnih operacija možemo definisati parcijalne klonove, NS klonove i hiperklonove kao skupove odgovarajućih operacija koji sadrže sve projekcije i zatvoreni su u odnosu na odgovarajuće kompozicije ili kao poduniverzume odgovarajućih algebri. Takođe su mreža parcijalnih klonova $\mathcal{L}_k^p$, mreža NS klonova $\mathcal{L}_k^{\text{IS}}$ i mreža hiperklonova $\mathcal{L}_k^h$ na E_k algebarske mreže.

Mreže

Moglo bi se reći da je predmet istraživanja teorije klonova opis mreže klonova na datom skupu. Na dvoelementnom skupu ova mreža je potpuno opisana zahvaljujući Emilu Postu [49]. On je pokazao da mreža $\mathcal{L}_2$ ima prebrojivo mnogo elemenata, koji su svi konačno generisani. Specijalno ima 7 atoma (klonovi koji su neposredno iznad J_2) i 5 koatoma (klonovi koji su neposredno ispod O_2). Njena struktura nije jako komplikovana i beskonačna je samo zato što sadrži 8 beskonačnih lanaca. Međutim, čim pređemo na skupove sa bar tri elementa stvari se bitno komplikuju jer je u tom slučaju mreža $\mathcal{L}_k$ neprebrojiva [75], i iako postoji dosta parcijalnih rezultata zapravo se veoma malo zna o njenoj strukturi.

Mreže parcijalnih klonova, NS klonova i hiperklonova su već na dvoelementnom skupu kardinalnosti kontinuum i stoga mnogo kompleksnije od $\mathcal{L}_2$. Za mrežu parcijalnih klonova $\mathcal{L}_2^p$ se recimo zna da ima 11 atoma [9] i 8 koatoma [26], i da sadrži elemente koji nisu konačno generisani. Takođe je poznato za koje totalne klonove C je interval $\mathcal{I}(C)$ (skup svih parcijalnih klonova čiji totalni deo je klon C) konačan, dok je za sve ostale neprebrojiv [39, 20]. Mreže $\mathcal{L}_2^{\mathrm{IS}}$ i $\mathcal{L}_2^h$ su izomorfne i sadrže 13 atoma [45] i 9 koatoma [71].

Potapanja

Logično je zapitati se kako su gore pomenute mreže međusobno povezane i da li se neke od osobina mreže totalnih klonova, koja je za sada najviše proučavana, mogu preneti na mreže $\mathcal{L}_k^p$, $\mathcal{L}_k^{\mathrm{IS}}$ i $\mathcal{L}_k^h$. Moguće je svakoj parcijalnoj operaciji, NS operaciji i hiperoperaciji pridružiti odgovarajuću totalnu operaciju, i ta preslikavanja nam, uz izvesne modifikacije, daju potapanja mreža $\mathcal{L}_k^p$, $\mathcal{L}_k^{\mathrm{IS}}$ i $\mathcal{L}_k^h$ u odgovarajuće mreže totalnih operacija.

Parcijalnoj operaciji $f \in P_k$ dodeljujemo operaciju $f_+ \in O_{k+1}$ na sledeći način

$$f_+(\vec{x}) = \begin{cases} f(\vec{x}), & \text{ako } \vec{x} \in E_k^n, \\ k, & \text{inače,} \end{cases}$$

za sve $\vec{x} \in E_{k+1}^n$. Za $F \subseteq P_k$ je $F_+ = \{f_+ : f \in F\}$.

Međutim proširenje parcijalnog klona na E_k nije klon na E_{k+1} jer ne sadrži projekcije.

Definišimo preslikavanje $g \mapsto g_-$ iz O_{k+1} u P_k sa $g_-(\vec{x}) = g(\vec{x})$, $\vec{x} \in E_k^n$. Za $G \subseteq O_{k+1}$ je $G_- = \{g_- : g \in G\}$. Sada možemo pokazati da je mreža parcijalnih klonova na E_k izomorfna podmreži mreže klonova na O_{k+1}. Preciznije, preslikavanje dato sa $G \mapsto G_-$ je izomorfizam između mreža $\mathcal{L}(\langle (J_k)_+ \rangle; \langle (H_k)_+ \rangle)$ i $\mathcal{L}_k^p$ [58, 6, 7].

Svakoj NS operaciji f na E_k možemo pridružiti operaciju f^+ na E_{k+1} na sledeći način

$$f^+(\vec{x}) = \prod \left\{ f(\vec{y}) : \vec{y} \in E_k^n \text{ i } \vec{y} \sqsubseteq \vec{x} \right\},$$

za sve $\vec{x} \in E_{k+1}^n$. Preslikavanje f^+ naziva se proširena NS operacija. Skup svih proširenih NS operacija iz $F \subseteq I_k$ označićemo sa F^+.

Ako je C klon nepotpuno specificiranih operacija na E_k, onda C^+ nije obavezno klon na E_{k+1} jer kompozicija proširenih NS operacija ne mora biti proširena NS operacija. I u slučaju Mal'tsevljevih operacija, skup svih proširenih NS operacija I_k^+ nije zatvoren u odnosu na Δ, za $k \geq 2$, i nije zatvoren u odnosu na $*$, za $k \geq 3$.

Kako bismo dobili algebru proširenih NS operacija koja je izomorfna punoj algebri NS operacija, definišemo sledeće operacije na I_k^+ :

$$\Delta_i : (I_k^+)^{(n)} \to (I_k^+)^{(n-1)}, \quad f^+ \mapsto (\Delta f)^+ \text{ i}$$

$$*_i : (I_k^+)^{(n)} \times (I_k^+)^{(m)} \to (I_k^+)^{(m)}, \quad (f^+, g^+) \mapsto (f \diamond g)^+.$$

Sada je preslikavanje $f \mapsto f^+$ izomorfizam između algebri $\mathcal{I}_k = (I_k; \diamond, \zeta, \tau, \Delta, e_1^{2,E_k})$ i $\mathcal{I}_k^+ = (I_k^+; *_i, \zeta, \tau, \Delta_i, e_1^{2,E_{k+1}})$ [16].

Proizvoljnoj hiperoperaciji $f \in H_k^{(n)}$ možemo dodeliti n-arnu operaciju $f^\#$ na $P_{E_k}^*$ datu sa

$$f^\#(X_1, \ldots, X_n) = \bigcup \left\{ f(x_1, \ldots, x_n) \,:\, x_i \in X_i, 1 \leq i \leq n \right\},$$

za sve $X_1, \ldots, X_n \in P_{E_k}^*$. Za skup $F \subseteq H_k$ označimo $F^\# = \{f^\# : f \in F\}$.

Nažalost, kao i u prethodna dva slučaja, pridruživanje skupa $C^\#$ nekom hiperklonu C nije željeno potapanje mreže hiperklonova na E_k u mrežu klonova na $P_{E_k}^*$, s obzirom da iz činjenice da je C hiperklon na E_k ne sledi da je $C^\#$ klon na $P_{E_k}^*$ jer kompozicija nije saglasna sa operatorom $\#$.

Zato ćemo hiperklonu C umesto skupa $C^\#$ dodeliti klon $\langle C^\# \rangle_{P_{E_k}^*}$. Šta više, ispostavlja se da se najmanji klon koji sadrži $C^{\#}$ može dobiti od $C^{\#}$ samo primenom transformacija mesta promenljivih, te za hiperklon C na E_k važi $\langle C^{\#}\rangle_{P^*_{E_k}} = \delta(C^{\#})$ [10]. Sada možemo definisati preslikavanje

$$\lambda : L^h_A \to L_{P^*_A}, \ \ C \mapsto \delta(C^{\#}).$$

Ovo preslikavanje je monotono potapanje, ali važi

$$(\exists C_1, C_2 \in L^h_k)\ [\lambda(C_1), \lambda(C_2)] \setminus im\lambda \neq \emptyset.$$

Takođe, saglasno je sa operacijom $\wedge$, ali ne i sa operacijom $\vee$ [23].

S druge strane, operator $\#$ se slaže sa Mal'tsevljevim operacijama $\circ, \zeta$ i τ, ali skup svih proširenih hiperoperacija $H^{\#}_k$ nije zatvoren u odnosu na Δ. Međutim, ako definišemo sledeću operaciju na $H^{\#}_k$:

$$\Delta_h : (H^{\#}_k)^{(n)} \to (H^{\#}_k)^{(n-1)}, \quad f^{\#} \mapsto (\Delta f)^{\#},$$

onda je preslikavanje $f \mapsto f^{\#}$ izomorfizam između algebri $\mathcal{H}_k = (H_k; \circ, \zeta, \tau, \Delta, e_1^{2,E_k})$ i $\mathcal{H}^{\#}_k = (H^{\#}_k; *, \zeta, \tau, \Delta_h, e_1^{2,P^*_k})$ [59].

Polimorfizmi

Iako postoje različiti pristupi u proučavanju mreže klonova, možda najmoćniji alat predstavlja Galoisova veza između skupa operacija O_k i skupa relacija $R_k = \bigcup\limits_{\ell \geq 1} \mathcal{P}(E^{\ell}_k)$. Do nje dolazimo na sledeći način.

Neka su $\rho \in R^{(\ell)}_k$ i $f \in O^{(n)}_k$. Za operaciju f kažemo da čuva relaciju ρ, odnosno relacija ρ je saglasna sa operacijom f ako važi

$$\begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{\ell 1} \end{pmatrix}, \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{\ell 2} \end{pmatrix}, \ldots, \begin{pmatrix} a_{1n} \\ a_{21} \\ \vdots \\ a_{\ell n} \end{pmatrix} \in \rho \quad \Rightarrow \quad \begin{pmatrix} f(a_{11}, a_{12}, \ldots, a_{1n}) \\ f(a_{21}, a_{22}, \ldots, a_{2n}) \\ \vdots \\ f(a_{\ell 1}, a_{\ell 2}, \ldots, a_{\ell n}) \end{pmatrix} \in \rho.$$

Još kažemo da je f polimorfizam od ρ. Skup svih operacija koje čuvaju ρ označavamo sa $Pol\,\rho$, a skup svih relacija saglasnih sa f sa $Inv\,f$. Ako definišemo preslikavanja

$$Pol : \mathcal{P}(R_k) \to \mathcal{P}(O_k) \quad \text{i} \quad Inv : \mathcal{P}(O_k) \to \mathcal{P}(R_k)$$

sa

$$Pol\, Q = \bigcap_{\rho \in Q} Pol\, \rho = \{f \in O_k : f \text{ čuva svako } \rho \in Q\},\ Q \subseteq R_k,$$

$$Inv\, F = \bigcap_{f \in F} Inv\, f = \{\rho \in R_k : \text{svako } f \in F \text{ čuva } \rho\},\ F \subseteq O_k.$$

Lako se vidi da je par (Pol, Inv) Galoisova veza između operacija i relacija.

Jednostavno se pokazuje da je $Pol\, Q$ klon, za svaki skup relacija Q, i da je $Inv\, F$ relacijski klon za svaki skup operacija F. Međutim važi i obratno, odnosno svaki klon je skup polimorfizama nekog skupa relacija i svaki relacijski klon je oblika $Inv\, F$ za neko $F \subseteq O_k$. Tačnije za klon C imamo $C = Pol(Inv\, C)$ dok za relacijski klon Q važi $Q = Inv(Pol\, Q)$ [4, 27].

Od velike pomoći pri proučavanju mreže klonova je činjenica koja je direktna posledica definicije Galoisove veze, a to je da što je veći klon to je manji odgovarajući skup relacija. Specijalno, što ćemo kasnije videti, svaki maksimalni klon je skup polimorfizama jedne relacije.

U slučaju parcijalnih operacija posmatramo relacije na skupu E_{k+1}. Za parcijalnu operaciju $f \in P_k^{(n)}$ kažemo da čuva relaciju $\rho \in R_{k+1}^{(\ell)}$ (odnosno, relacija ρ je saglasna sa f) ako za svaku matricu M čije su kolone u ρ važi $f_+(M) \in \rho$.

Sa $pPol\, \rho$ označavamo skup svih parcijalnih operacija koje čuvaju relaciju ρ, a sa $pInv\, f$ skup svih relacija koje su saglasne sa f. Takođe uvodimo oznaku

$$pPOL\, \rho = pPol\big(\rho \cup (E_{k+1}^{\ell} \setminus E_k^{\ell})\big),$$

pri čemu je relacija $\rho \cup (E_{k+1}^{\ell} \setminus E_k^{\ell})$ puno proširenje relacije ρ. Može se pokazati da su za proizvoljno $\rho \in R_{k+1}^{(\ell)}$ skupovi $pPol\, \rho$ i $pPOL\, \rho$ parcijalni klonovi. Šta više, $pPOL\, \rho$ je jak parcijalni klon, tj. parcijalni klon koji sadrži sve podfunkcije svojih elemenata.

Definišimo preslikavanja

$$pPol : \mathcal{P}(R_{k+1}) \to \mathcal{P}(P_k) \quad \text{i} \quad pInv : \mathcal{P}(P_k) \to \mathcal{P}(R_{k+1})$$

na sledeći način

$$pPol\,Q = \bigcap_{\rho\in Q} pPol\,\rho = \{f \in P_k : f \text{ čuva svaku } \rho \in Q\},\ Q \subseteq R_{k+1},$$

$$pInv\,F = \bigcap_{f\in F} pInv\,f = \{\rho \in R_{k+1} : \text{svako } f \in F \text{ čuva } \rho\},\ F \subseteq P_k.$$

Očigledno je par $(pPol, pInv)$ Galoisova veza između relacija i parcijalnih operacija.

Ako primenimo hiperoperaciju na matricu čije su kolone elementi neke ℓ-arne relacije, kao rezultat dobijamo ℓ-torku skupova, pa postoji više načina da se definiše svojstvo saglasnosti relacije sa hiperoperacijom. Ovde ćemo predstaviti dve relacije na $P^*_{E_k}$ pomoću kojih ćemo definisati ovu saglasnost, kao i Galoisove veze koje oni indukuju.

Neka je $\ell \geq 1$ i $\rho \in R_k^{(\ell)}$. Jako proširenje relacije ρ je relacija ρ_d definisana sa

$$\rho_d = \big\{(A_1, \dots, A_\ell) \in (P^*_{E_k})^\ell : A_1 \times \cdots \times A_\ell \subseteq \rho\big\}.$$

Ovo znači da je $(A_1, \dots, A_\ell)$ u ρ_d ako je svaka ℓ-torka $(a_1, \dots, a_\ell) \in A_1 \times \cdots \times A_\ell$ sadržana u ρ. Kažemo da hiperoperacija $f \in H_k^{(n)}$ d-čuva relaciju $\rho \in R_k^{(\ell)}$ ako za svaku $\ell \times n$ matricu M čije su kolone u ρ važi $f(M) \in \rho_d$, tj. $A_1 \times \cdots \times A_\ell \subseteq \rho$. Skup svih hiperoperacija koje d-čuvaju relaciju ρ označavamo sa $dPol\,\rho$, a skup svih relacija koje d-čuva hiperoperacija f sa $dInv\,f$. Galoisovu vezu $(dPol\,\rho, dInv\,f)$ definišemo slično kao u prethodnim slučajevima.

Za svako $Q \subseteq R_k$ skup $dPol\,Q$ je hiperklon i to hiperklon koji sadrži sve pod-hiperoperacije svojih elemenata. Takve hiperklonove nazivamo nadole zatvoreni hiperklonovi [8, 53, 54].

S druge strane, za ℓ-arnu relaciju ρ na E_k možemo definisati slabo proširenje ρ_h na sledeći način

$$\rho_h = \big\{(A_1, \dots, A_\ell) \in (P^*_{E_k})^\ell : (A_1 \times \ldots \times A_\ell) \cap \rho \neq \emptyset\big\},$$

tj. ρ_h se sastoji od ℓ-torki $(A_1, \dots, A_\ell)$ podskupova od E_k takvih da postoji $(a_1, \dots, a_\ell) \in A_1 \times \cdots \times A_\ell$ koja je u ρ [61]. Hiperoperacija $f \in H_k^{(n)}$ h-čuva relaciju $\rho \in R_k^{(\ell)}$ ako za svaku $\ell \times n$ matricu M čije su kolone u

ρ važi $f(M) \in \rho_h$. Skup $hPol\,\rho$ se sastoji od svih hiperoperacija koje h-čuvaju relaciju ρ, a $hInv\,f$ je skup svih relacija koje h-čuva hiperoperacija f. Ponovo, Galoisova veza $(hPol\,\rho, hInv\,f)$ se definiše analogno prethodnim slučajevima.

Skup $hPol\,Q$ je hiperklon za svaki skup relacija Q, i ima osobinu da sadrži sve nad-hiperoperacije svojih elemenata. Takve hiperklonove nazivamo prema gore zatvoreni hiperklonovi [17].

Svojstvo saglasnosti relacije i NS operacije možemo definisati slično kao u slučaju hiperoperacija, pri čemu je jako proširenje relacije ρ dato sa

$$\rho_s = \Big\{(a_1, \ldots, a_\ell) \in E_{k+1}^\ell : \big(\forall (b_1, \ldots, b_\ell) \in E_k^\ell\big)\, (b_1, \ldots, b_\ell) \sqsubseteq (a_1, \ldots, a_\ell) \Rightarrow (b_1, \ldots, b_\ell) \in \rho\Big\},$$

dok je slabo proširenje definisano sa

$$\rho_w = \Big\{(a_1, \ldots, a_\ell) \in E_{k+1}^\ell : \big(\exists (b_1, \ldots, b_\ell) \in \rho\big)\, (b_1, \ldots, b_\ell) \sqsubseteq (a_1, \ldots, a_\ell)\Big\}.$$

Koatomi

Na kraju ćemo predstaviti rezultate vezane za koatome mreža $\mathcal{L}_k$, $\mathcal{L}_k^p$, $\mathcal{L}_k^h$ i $\mathcal{L}_k^{\mathrm{IS}}$. Veoma je bitno opisati maksimalne elemente mreže jer se oni koriste pri formulisanju kriterijuma kompletnosti. Kažemo da je skup $F \subseteq O_k$ kompletan ako generiše ceo skup O_k, tj. $\langle F \rangle = O_k$. Poznato je da postoji konačno mnogo maksimalnih klonova na E_k i svaki pravi potklon od O_k je sadržan u nekom maksimalnom klonu [74]. Stoga je skup $F \subseteq O_k$ kompletan ako i samo ako nije podskup nijednog maksimalnog klona. Isto važi i za mreže parcijalnih klonova, NS klonova i hiperklonova.

Kriterijumi kompletnosti za totalne i parcijalne klonove su određeni, ali smo još uvek daleko od dostizanja takvog rezultata u slučaju hiperklonova i NS klonova. Ipak poznate su neke klase maksimalnih klonova hiperoperacija i NS operacija.

Jedno od najznačajnijih dostignuća u teoriji klonova do sada je kompletna klasifikacija koatoma mreže $\mathcal{L}_k$ do koje je došao I.G. Rosenberg. Iako je ovo veličanstven rezultat, moramo reći da on predstavlja kulminaciju udruženih napora više matematičara koji su se 50tih i 60tih godina prošlog veka bavili problemom opisivanja svih maksimalnih klonova na datom skupu.

Jablonski je u [74] pokazao jedan specijalni slučaj, da je $Pol\,\iota_k^n$, gde je $\iota_k^n = \big\{(x_1,\ldots,x_n) \in E_k^n \;:\; |\{x_1,\ldots,x_n\}| \leq n-1\big\}$, maksimalan klon, i to jedini maksimalan klon koji sadrži sve unarne operacije na E_k. Opštije, Kuznjecov je u [35] dokazao da je svaki maksimalan klon potpuno određen jednom relacijom, preciznije, svaki maksimalan klon je oblika $Pol\,\rho$ za neku nedijagonalnu relaciju ρ. Možemo se zapitati da li je ovo najpreciznija karakterizacija maksimalnih klonova koju možemo dobiti. Srećom odgovor je negativan.

Nakon Posta, koji je opisavši sve klonove na dvoelementnom skupu takođe naveo i 5 maksimalnih, Jablonski je u [73] odredio svih 18 maksimalnih klonova na E_3, a navodno je Mal'tsev pokazao da postoje tačno 82 maksimalna klona na E_4. Zatim je u [55] Rosenberg opisao šest klasa relacija koje određuju maksimalne klonove na proizvoljnom konačnom skupu, a konačno je u [56] dokazao da je ova lista potpuna.

Navedimo sada poznatu Rosenbergovu teoremu.

Klon na E_k je maksimalan ako i samo ako je oblika $Pol\,\rho$, gde je ρ jedna od sledećih relacija:

(R_1) ograničeno parcijalno uređenje;

(R_2) graf permutacije prostog reda;

(R_3) prosta afina relacija;

(R_4) netrivijalna relacija ekvivalencije;

(R_5) centralna relacija;

(R_6) ℓ-regularna relacija, $\ell \geq 3$.

Skup $O_k \cup \langle c_k \rangle_p$ čine sve totalne operacije na E_k i sve parcijalne operacije koje nisu definisane ni za jednu ulaznu vrednost. Ovaj skup je maksimalan parcijalni klon, i to je jedini maksimalni parcijalni klon koji sadrži O_k.

Problem kompletnosti za Bulove parcijalne operacije je rešio Freivald [26].

Isti problem na E_3 su nezavisno rešili Lau [36] i Romov [51], dok su za opis svih koatoma mreže parcijalnih klonova na proizvoljnom konačnom skupu zaslužni Haddad i Rosenberg [28, 29, 30]. Navešćemo njihovu klasifikaciju.

Parcijalni klon M na E_k je maksimalan ako i samo ako je $M = O_k \cup \langle c_k \rangle_p$ ili je oblika $M = pPOL\rho$, pri čemu je ρ jedna od sledećih relacija:

- *ℓ-arna ($1 \leq \ell \leq k$) netrivijalna totalno refleksivna i totalno simetrična relacija;*

- *ℓ-arna ($\ell \geq 2$) koherentna arefleksivna ili kvazi-dijagonalna relacija;*

- *kvatenarna koherentna relacija $\sigma \cup \rho_i$, $i \in \{1, 2\}$, gde je σ neprazna kvatenarna arefleksivna relacija i*

$$\rho_1 = \big\{(a, a, b, b), (a, b, a, b) \,:\, a, b \in E_k\big\},$$
$$\rho_2 = \big\{(a, a, b, b), (a, b, a, b), (a, b, b, a) \,:\, a, b \in E_k\big\}.$$

Skup svih totalnih operacija je koatom u mreži hiperklonova, što nije bio slučaj kod parcijalnih klonova. Dakle za svako $f \in H_k \setminus O_k$ imamo $\langle O_k \cup \{f\}\rangle_h = H_k$, pa pošto je O_k konačno generisan, zaključujemo da isto važi i za H_k.

Sada ćemo predstaviti četiri klase maksimalnih hiperklonova određenih nekim od Rosenbergovih relacija. Radi se o hiperklonovima oblika $hPol\rho$ i dovoljan uslov da takav hiperklon bude maksimalan je da je $Pol\rho$ maksimalan klon i važi

$$(\forall f \in H_k \setminus hPol\,\rho)\,(\exists f' \in O_k \setminus Pol\,\rho)\ f' \in \langle Pol\,\rho \cup \{f\}\rangle_h.$$

Tako je konstrukcijom odgovarajuće totalne operacije $f' \notin Pol\,\rho$ pomoću nekih operacija iz $Pol\,\rho$ i hiperoperacije $f \notin hPol\,\rho$ dokazano da je $hPol\,\rho$ maksimalan hiperklon u slučaju kada je ρ ograničeno parcijalno uređenje [17] i kada je ρ netrivijalna relacija ekvivalencije, centralna relacija ili regularna relacija [41].

Slično kao u slučaju hiperklonova možemo pokazati da je skup svih totalnih operacija maksimalan NS klon, pa je i I_k konačno generisan.

U nastavku razmatramo četiri klase relacija na E_k sa osobinom da je njihovo slabo proširenje jednako punom proširenju, a koje su takve da je $wPol\,\rho$

maksimalan klon nepotpuno specificiranih operacija. Ovi rezultati su analogni gore navedenim rezultatima za hiperklonove. Naime, i u ovom slučaju kako bismo pokazali da je $wPol\,\rho$ maksimalan NS klon ako je $Pol\,\rho$ maksimalan klon, određujemo totalnu operaciju $f' \notin Pol\,\rho$ koja je generisana operacijama iz $Pol\,\rho$ i NS operacijom $f \notin wPol\,\rho$. Lako se dokazuje da ako je relacija totalno refleksivna, onda je njeno slabo proširenje jednako sa punim proširenjem. Ovo važi za sve relacije iz Rosenbergovih klasa (R1), (R4), (R5) i (R6), pri čemu su relacije iz (R4), (R5) i (R6) istovremeno i totalno simetrične. Konstrukcija operacije f' je u slučaju NS klonova mnogo jednostavnija nego u slučaju hiperklonova, pogotovu za relacije iz (R1). Dakle, $wPol\,\rho$ je maksimalan NS klon u slučaju kada je ρ ograničeno parcijalno uređenje, netrivijalna relacija ekvivalencije, centralna ili regularna relacija.

Struktura rada

Ovaj rad se sastoji iz 7 poglavlja. **Prvo poglavlje** je uvodno, i u njemu su dati motivacija za istraživanja, kao i pregled rezultata po poglavljima, sa naglaskom na originalnim doprinosima.

U **drugom delu** dajemo pregled osnovnih definicija vezanih za klonove totalnih operacija, parcijalnih operacija, nepotpuno specificiranih operacija i hiperoperacija.

Originalni doprinos predstavlja deo o nepotpuno specificiranim operacijama, objavljen u radovima [15] i [16], gde formalno uvodimo pojam NS operacije, definišemo kompoziciju takvih operacija i njihove klonove, i navodimo osnovne osobine mreže NS klonova. Takođe definišemo tri unarne i jednu binarnu operaciju na skupu svih NS operacija, čime dobijamo punu algebru nepotpuno specificiranih operacija.

U **trećem poglavlju** je prikazano kako je moguće svakoj parcijalnoj operaciji, nepotpuno specificiranoj operaciji i hiperoperaciji na skupu E_k pridružiti totalnu operaciju na E_{k+1} u slučaju parcijalnih i NS operacija, odnosno na $P^*_{E_k}$ u slučaju hiperoperacija. Koristeći ova preslikavanja možemo dobiti potapanja mreža parcijalnih klonova, NS klonova i hiperklonova u odgovarajuće mreže totalnih klonova, što nam omogućava da izvesne osobine mreže totalnih klonova prenesemo na preostale tri mreže.

Originalni doprinos je deo o proširenju NS operacija objavljen u radu [16].

Kao što je rečeno, NS operacijama na E_k se dodeljuju odgovarajuće operacije na E_{k+1}. S obzirom da ovako dobijeno preslikavanje nije homomorfizam, uvodimo pogodne modifikacije Mal'tsevljevih operacije na skupu proširenih NS operacija kako bismo dobili algebru proširenih NS operacija izomorfnu punoj algebri NS operacija.

U **četvrtom delu** se bavimo poznatom Galoisovom vezom (Pol, Inv) između relacija i operacija, kao jednim od najvažnijih alata u proučavanju mreže klonova, i modifikacijama ove veze za parcijalne operacije, NS operacije i hiperoperacije.

Originalni doprinos je deo o prema gore zatvorenim hiperklonovima, objavljen u radu [17]. Dualno slučaju nadole zatvorenih hipeklonova, koji sadrže sve svoje pod-hiperoperacije, a koje su nezavisno proučavali Börner [8] i Romov [53, 54], možemo posmatrati hiperklonove koji sadrže sve nad-hiperoperacije svojih elemenata i njih zovemo prema gore zatvoreni hiperklonovi. Oni formiraju algebarsku mrežu u odnosu na skupovnu inkluziju. Potom opisujemo jednu klasu prema gore zatvorenih hiperklonova uvodeći odgovarajuću Galoisovu vezu između relacija i hiperklonova, indukovanu slabim čuvanjem relacije.

Originalni rezultati se nalaze i u sekciji o polimorfizmima u mreži nepotpuno specificiranih klonova, pri čemu je deo o vezi između proširenih NS operacija i relacija objavljen u radu [16]. Definišemo nadole zatvorene i prema gore zatvorene NS klonove i dve Galoisove veze između relacija i NS operacija analogno kao u slučaju hiperklonova. U poslednjem delu pokazujuemo da skup svih proširenih NS operacija koje čuvaju neku relaciju u opštem slučaju ne mora biti zatvoren u odnosu na modifikovane Mal'tsevljeve operacije definisane u prethodnom poglavlju, a onda predstavljamo dve klase relacija, koje odgovaraju jakom i slabom proširenju, takve da je skup proširenih NS operacija koje ih čuvaju zapravo proširenje nekog NS klona.

U **petom poglavlju** navodimo neke poznate rezultate vezane za mreže totalnih klonova, parcijalnih klonova i hiperklonova na dvoelementnom skupu. U ovom slučaju mreže hiperklonova i NS klonova su izomorfne.

Šesto poglavlje je posvećeno koatomima. Navodimo kriterijume kompletnosti za skupove totalnih i parcijalnih operacija, a zatim predstavljamo neke klase maksimalnih hiperklonova i nepotpuno specificiranih klonova.

Sekcija o hiperklonovima određenim ograničenim parcijalnim uređenjima objavljena je u radu [17]. Ovi rezultati se takođe pojavljuju u autorkinom master radu [13], s obirom da je to bio deo istraživanja u tom trenutku. Pokazujemo da je za svako ograničeno parcijalno uređenje ρ skup svih hiperoperacija koje slabo čuvaju relaciju ρ ($hPol\rho$) maksimalan hiperklon, tako što konstruišemo operaciju f' koja nije u $Pol\rho$ koristeći proizvoljnu hiperoperaciju $f \notin hPol\rho$ i neke operacije iz $Pol\rho$. Centralni deo je definicija pomoćne operacije $g^{\rho}_{b,c}$ i dokaz da ona čuva ρ.

Originalni doprinos je sekcija o maksimalnim NS klonovima. Najpre dokazujemo da je, kao i u slučaju hiperklonova, skup svih totalnih operacija maksimalan NS klon. U ostatku sekcije pokazujemo da ako je ρ ograničeno parcijalno uređenje, netrivijalna relacija ekvivalencije, centralna relacija ili regularna relacija, onda je skup svih NS operacija koje slabo čuvaju relaciju ρ ($wPol\rho$) maksimalan NS klon. Ono što omogućava da ovi dokazi budu znatno jednostavniji nego u slučaju hiperklonova je činjenica da su za relacije iz pomenutih klasa slabo proširenje i puno proširenje jednaki.

U **sedmom poglavlju** su predstavljene neke primene nedeterminizma i teorije klonova u teorijskom računarstvu, kao i pregled otvorenih problema koji bi mogli biti interesantni za dalje istraživanje.

Abstract

This thesis is a survey of some well known and several new results concerning lattices of total clones, partial clones, incompletely specified clones and hyperclones. We assign to every partial operation, incompletely specified operation and hyperoperation a suitable total operation and investigate thereby induced embeddings of the three lattices into corresponding lattices of total clones. Next we modify the famous Galois connection (Pol, Inv) between relations and operations for partial operations, IS operations and hyperoperations. In the latter two cases we analogously describe certain classes of clones which correlate to two of the possible ways to define a preservation property. We also state some known results concerning the four lattices on a two-element set. Finally, we present completeness criteria for the lattices of total and partial clones, and in the case of hyperclones and incompletely specified clones we describe four classes of coatoms, determined by four classes of Rosenberg's relations.

Contents

Chapter 1

Introduction

If an operation does not provide output values for all of the input values, we say that it is partially defined. For a nonempty set A, and D a proper subset of A^n, such an operation f is a mapping from D into A. But what if we do not consider $f(a_1, \ldots, a_n)$, where $(a_1, \ldots, a_n) \in A^n \setminus D$, to be undefined, but rather unspecified? This assumption yields significant difference when it comes to the composition of such operations.

Let $A = \{0, 1\}$ and $\texttt{AND}$ be the ordinary conjunction on A. Suppose f and g are unary operations on A for which $f(0) = 0$ and the value $g(0)$ is not specified. Then the (output) value of the composition $h(x) = \texttt{AND}(f(x), g(x))$ for $x = 0$ is specified as 0, i.e., we have

$$h(0) = \texttt{AND}(f(0), g(0)) = \texttt{AND}(0, g(0)) = 0.$$

This makes perfect sense since the binary operation $\texttt{AND}$ takes the value 0 whenever at least one of the arguments is 0, regardless of the value of the other argument. However, operation h would not be defined on $\{0\}$ if we take $g(0)$ to be undefined, in other words if operations and composition are considered in the setting of partial operations and their standard composition.

How can we interpret those unspecified outputs? One possibility is that we have different outputs for the same input value, i.e., the output is a nonempty subset of A, which gives us hyperoperations. On the other hand, we may take some $u \notin A$, and regard it as any one of the values from A, which gives us incompletely specified operations.

On a two-element set, these two concepts are basicaly the same, since $|A \cup \{u\}| = |\mathcal{P}(A) \setminus \{\emptyset\}| = 3$. Nevertheless, it is evident that for $|A| > 2$ hyperoperations have more possible output values than incompletely specified operations.

We can use both hyperoperations and incompletely specified operations for modelling nondeterministic processes.

For instance, in software systems, repeated executions of a program may produce different results, or concurrent processes may have different reductions. This behaviour can be formally modelled as a function that assigns a set of values to a given argument. These functions are basically hyperoperations. A widely studied example where such a function also appears is the transition function in the definition of nondeterministic finite automata.

On the other hand, in optimization of logic circuits, input assignments, for which output is not specified, are called "don't care" conditions and they play important role in determination of minimal disjunctive normal forms and design of equivalent logic circuits.

$* \quad * \quad * \quad * \quad *$

This thesis presents a comparative study of the lattices of total clones, partial clones, incompletely specified clones and hyperclones.

For more details about clone theory we refer the reader to books by Pöschel and Kalužnin [48], Szendrei [70] and Lau [38]. The last one also covers results concerning partial clones to some extent.

Instigated by the lack of coherent theory for general hyperstructure, Rosenberg introduced the notion of a hyperclone in [59] and [60], while paper [18] summarises different approaches to the study of the hyperclone lattice.

Investigation of incompletely specified operations was initiated by the Kleene's three-value logic [34] and a non-standard composition of Boolean partial operations, introduced by Tarasov in [71].

Chapter overview

In **Chapter 2** we introduce clones of total operations, partial operations, incompletely specified operations and hyperoperations, by defining each of them both as the composition closed set of operations that contains all projections and as subuniverse of a certain algebra.

Original contribution is Section 2.3 about incompletely specified operations, published in [15] and [16]. Here we formally introduce the notion of an IS operation, then we define the composition of such operations and consequently give a definition of an IS clone. We also state some basic properties of the lattice of IS clones. Analogue to the total, partial and hyper case we define three unary and one binary operation (called Mal'tsev operations) on the set of all IS operations, thereby obtaining full algebra of incompletely specified operations.

In **Chapter 3** we assign to every partial operation, incompletely specified operation and hyperoperation a suitable total operation. Using these mappings we can induce embeddings of lattices of partial clones, IS clones and hyperclones into corresponding lattices of total clones, which enables us to transfer certain properties of the (total) clone lattice onto the other three lattices.

Original contribution is the Section 3.2 on one-point extension of IS operations, published in [16]. As we said, to each IS operation on E_k a corresponding operation on E_{k+1} is assigned. Since the induced mapping is not homomorphic, we modify Mal'tsev operations on the set of extended IS operations I_k^+ in order to get algebra of extended IS operations isomorphic to the full algebra of IS operations.

In **Chapter 4** we deal with the famous Galois connection (Pol, Inv) between relations and operations, as one of the fundamental tools used in the investigation of the clone lattice, and we suggest some modifications of this connection for partial operations, IS operations and hyperoperations.

Original contribution is in the Section 4.4.2 about upward saturated hyperclones, published in [17]. Dually to the case of down closed hyperclones, that is, hyperclones containing all sub-hyperoperations of their elements, which were independently studied by Börner [8] and Romov [53, 54], we can consider hyperclones containing all super-hyperoperations of their elements, and we call them upper saturated hyperclones. The set of all such hyperclones forms an algebraic lattice with respect to set inclusion. Next we describe one class of upper saturated hyperclones by introducing Galois connection between relations and hyperoperations induced by weak preservation of a relation.

Original results also appear in Section 4.5 on polymorphisms in the lattice of incompletely specified clones. Analogous to the case of hyperclones we define down closed and upper saturated IS clones and two corresponding Galois connections between relations and IS operations. Section 4.5.3 about the connection between extended IS operations and relations is published in [16]. Here we show that the set of all extended IS operations that preserve some relation does not necessarily have to be closed under modified Mal'tsev operations defined in the previous chapter. Then we present two classes of relations on E_{k+1}, corresponding to strong and weak extension, such that the set of extended IS operations preserving them is actually extension of some IS clone.

In **Chapter 5** we state some known results concerning lattices of total clones, partial clones and hyperclones on a two-element set. As we already mentioned, in this case lattices of hyperclones and IS clones are isomorphic.

Chapter 6 is dedicated to coatoms. We state completeness criteria for the lattices of total and partial clones, and then present some classes of maximal hyperclones and maximal IS clones.

Section 6.3.1 about hyperclones determined by bounded partial orders is published in [17]. These results also appeared in author's Master thesis [13], as it was the part of the ongoing research at that time. We show that for every bounded partial order ρ set of all hyperoperations that weakly preserve relation ρ ($hPol\rho$) is a maximal hyperclone, by constructing an operation f' not in $Pol\rho$ using an arbitrary hyperoperation $f \notin hPol\rho$ and some operations from $Pol\rho$. The central part is definition of the auxiliary operation $g^{\rho}_{b,c}$ and the proof that it preserves relation ρ.

Original contribution is Section 6.4 about maximal IS clones. First we prove that, as in the case of hyperclones, the set of all total operations is a maximal IS clone. In the remainder of this section we show that if ρ is a bounded partial order, nontrivial equivalence relation, central relation or regular relation, set of all IS operations that weakly preserve relation ρ ($wPol\rho$) is a maximal IS clone. What makes these proofs considerably simpler than in the case of hyperclones is the fact that for relations from the afore mentioned classes weak extension and full extension coincide.

In **Chapter 7** we present certain applications of nondeterminism and clone theory in theoretical computer science, as well as some open problems that could be interesting for future investigation.

Chapter 2

Clones

Investigation of the clones in universal algebra is motivated by the fact that the set of term operations of an algebra $\mathbf{A} = (A, F)$ is always a clone. As a matter of fact, a set C of finitary operations on A is a clone if and only if there exists an algebra $\mathbf{A} = (A, F)$ such that C is the set of term operations of $\mathbf{A}$. Nevertheless, in the first section of this chapter two other equivalent definitions of a clone are given: as a composition-closed set of operations containing all projections and as subuniverse of a certain algebra. We also state that the set of all clones on given domain A ordered by set inclusion forms an algebraic lattice. Then in the remainder of the chapter we present similar characterizations for partial clones, IS clones and hyperclones.

In this thesis our domain will be the set $E_k = \{0, 1, \ldots, k-1\}$, with $k \geq 2$.

2.1 Clones of operations

A function $f : E_k^n \to E_k$ is an n-ary operation on E_k. We will denote by $O_k^{(n)} = E_k^{E_k^n}$ the set of all n-ary operations on E_k, and by $O_k = \bigcup_{n \geq 1} O_k^{(n)}$ the set of all finitary operations on E_k. For $F \subseteq O_k$ let $F^{(n)} = F \cap O_k^{(n)}$.

We specify some operations that will also appear later in the text.

Definition 2.1.1

(i) A constant operation *is an operation c_a^n, defined by*

$$c_a^n(x_1, \ldots, x_n) = a, \ a \in E_k.$$

(ii) For $i \in \{1, \dots, n\}$, i-th n-ary projection e_i^n *is defined by*

$$e_i^n(x_1, \dots, x_i, \dots, x_n) = x_i.$$

We denote by J_k *the set of all projections on* E_k.

(iii) A ternary majority operation, $\mathtt{ma} \in O_k^{(3)}$, *is such that*

$$\mathtt{ma}(x, x, y) = \mathtt{ma}(x, y, x) = \mathtt{ma}(y, x, x) = x, \text{ for all } x, y \in E_k;$$

(iv) A ternary minority operation, $\mathtt{mi} \in O_k^{(3)}$, *satisfies*

$$\mathtt{mi}(x, x, y) = \mathtt{mi}(x, y, x) = \mathtt{mi}(y, x, x) = y, \text{ for all } x, y \in E_k.$$

Definition 2.1.2 Composition *of operations* $f \in O_k^{(n)}$ *and* $g_1, \dots, g_n \in O_k^{(m)}$ *is the* m*-ary operation* $f(g_1, \dots, g_n) \in O_k^{(m)}$, *defined as follows*

$$\big(f(g_1, \dots, g_n)\big)(\vec{x}) = f\big(g_1(\vec{x}), \dots, g_n(\vec{x})\big),$$

where $\vec{x} = (x_1, \dots, x_m) \in E_k^m$.

Example 2.1.3 *In Figure 2.1 we see a binary operation* f *and two unary operations* g_1, g_2*, as well as their composition* $f(g_1, g_2)$*. For instance, we have*

$$\big(f(g_1, g_2)\big)(1) = f\big(g_1(1), g_2(1)\big) = f(2, 0) = 0.$$

f	0	1	2
0	0	1	2
1	0	1	1
2	0	2	0

	g_1	g_2
0	0	1
1	2	0
2	1	2

	$f(g_1, g_2)$
0	1
1	0
2	1

Figure 2.1: The composition of a binary and two unary operations on E_3.

Definition 2.1.4 *Set* $C \subseteq O_k$ *is called* clone of operations *on* E_k *if the following two conditions are satisfied:*

(i) C *contains all projections and*

(ii) C *is closed with respect to composition.*

Example 2.1.5 *The following sets of operations on E_k are clones*

(i) O_k *- set of all operations;*

(ii) J_k *- set of all projections;*

(iii) set of all idempotent operations ($f \in O_k$ *is idempotent if* $f(x, \ldots, x) = x$*, for all* $x \in E_k$*).*

Evidently, it holds that the intersection of an arbitrary family of clones is also a clone.

For a set $F \subseteq O_k$ the least clone containing F will be denoted by $\langle F \rangle$ and we have

$$\langle F \rangle = \bigcap \big\{ C \subseteq O_k : C \text{ is a clone and } F \subseteq C \big\}.$$

We say that $\langle F \rangle$ is a clone generated by F. Whenever F is a finite set, that is, if $F = \{f_1, \ldots, f_n\}$, we will write $\langle f_1, \ldots, f_n \rangle$ instead of $\langle \{f_1, \ldots, f_n\} \rangle$.

It is easy to prove that $\langle\ \rangle : \mathcal{P}(O_k) \to \mathcal{P}(O_k)$ is an algebraic closure operator, which directly implies the following theorem.

Theorem 2.1.6 *Clones of operations on a finite set E_k form an algebraic lattice $\mathcal{L}_k$ with respect to the set inclusion. The least element of the lattice is J_k, and the greatest element is O_k. Lattice operations are defined as follows*

$$C_1 \wedge C_2 = C_1 \cap C_2 \qquad \textit{and} \qquad C_1 \vee C_2 = \langle C_1 \cup C_2 \rangle.$$

As a consequence of the previous theorem, a clone can also be defined as a subuniverse of some algebra. In what follows we introduce one such algebra. Let ζ, τ, Δ be unary and $*$ binary operation on O_k. These operations are also called Mal'tsev operations. For the sake of simplicity we shall write $\zeta f, \tau f, \Delta f$ and $f * g$ instead of $\zeta(f), \tau(f), \Delta(f)$ and $*(f, g)$.

- For $f \in O_k^{(1)}$ let $\zeta f = \tau f = \Delta f = f$;
- for $f \in O_k^{(n)}, n \geq 2$, let $\zeta f, \tau f \in O_k^{(n)}$ and $\Delta f \in O_k^{(n-1)}$ be defined as

$$\begin{aligned}
&(\zeta f)(x_1, x_2, \ldots, x_n) = f(x_2, \ldots, x_n, x_1),\\
&(\tau f)(x_1, x_2, x_3, \ldots, x_n) = f(x_2, x_1, x_3, \ldots, x_n),\\
&(\Delta f)(x_1, x_2, \ldots, x_{n-1}) = f(x_1, x_1, x_2, \ldots, x_{n-1});
\end{aligned}$$

- for $f \in O_k^{(n)}$ and $g \in O_k^{(m)}$ let $f * g \in O_k^{(m+n-1)}$ be defined as

$$(f * g)(x_1, \ldots, x_{m+n-1}) = f\big(g(x_1, \ldots, x_m), x_{m+1}, \ldots, x_{m+n-1}\big).$$

With the addition of the first binary projection e_1^2, we obtain the algebra $\mathcal{O}_k = (O_k; *, \zeta, \tau, \Delta, e_1^2)$, called the *full algebra of operations*.

Theorem 2.1.7 *A set $C \subseteq O_k$ is a clone if and only if it is a subuniverse of the algebra $\mathcal{O}_k$.*

Sketch of a proof. If $C \subseteq O_k$ is a clone, then $J_k \subseteq C$, and thus $e_1^2 \in C$. Since C is also closed with respect to composition of operations, for $f \in C^{(n)}$ and $g \in C^{(m)}$ we obtain

$$\begin{aligned}
&\zeta f = f(e_2^n, \ldots, e_n^n, e_1^n),\\
&\tau f = f(e_2^n, e_1^n, e_3^n, \ldots, e_n^n),\\
&\Delta f = f(e_1^{n-1}, e_1^{n-1}, e_2^{n-1}, \ldots, e_{n-1}^{n-1}) \quad \text{and}\\
&f * g = f\big(g(e_1^{m+n-1}, \ldots, e_m^{m+n-1}), e_{m+1}^{m+n-1}, \ldots, e_{m+n-1}^{m+n-1}\big).
\end{aligned}$$

On the other hand, if C is a subuniverse of the algebra $\mathcal{O}_k$, then it is closed with respect to ζ, τ, Δ and $*$, and contains e_1^2. We can make an arbitrary projection in the following way

$$e_1^1 = \Delta e_1^2 \quad \text{and} \quad e_i^n = \zeta^{n-i} \nabla^{n-2} \tau e_1^2,$$

where ∇ is adding of a fictitious variable, i.e., $(\nabla f)(x_1, x_2, \ldots, x_{n+1}) = f(x_2, \ldots, x_{n+1})$ and it holds $\nabla f = f * (\tau e_1^2)$. Hence, $J_k \subseteq C$. Furthermore, composition $f(g_1, \ldots, g_n)$ of operations $f \in O_k^{(n)}$ and $g_1, \ldots, g_n \in O_k^{(m)}$ can be expressed by $*, \Delta$ and permuting variables. Therefore, C is a clone.

2.2 Clones of partial operations

A mapping from E_k^n to E_{k+1} is said to be a *partial operation* on E_k, if k is regarded as undefined. If we denote by $P_k^{(n)}$ the set of all n-ary partial operations on E_k, then $P_k = \bigcup_{n \geq 1} P_k^{(n)}$ will be the set of all partial operations on E_k. For $F \subseteq P_k$ let $F^{(n)} = F \cap P_k^{(n)}$.

Let $f \in P_k^{(n)}$ and $g_1, \dots, g_n \in P_k^{(m)}$. Composition of partial operations $f, g_1, \dots, g_n$ is an m-ary partial operation $f(g_1, \dots, g_n)$, defined as

$$\big(f(g_1, \dots, g_n)\big)(\vec{x}) = \begin{cases} f\big(g_1(\vec{x}), \dots, g_n(\vec{x})\big), & \text{if } g_i(\vec{x}) \in E_k,\ 1 \le i \le n, \\ k, & \text{otherwise}, \end{cases} \tag{2.1}$$

where $\vec{x} = (x_1, \dots, x_m) \in E_k^m$.

This means that whenever at least one of the partial operations $f, g_1, \dots, g_n$ is undefined, the whole composition is going to be undefined, as shown in the following example.

Example 2.2.1 *In the Figure 2.2 we have one binary (f) and two unary (g_1, g_2) partial operations on E_3, and their composition $f(g_1, g_2)$. For example we have*

$$\big(f(g_1, g_2)\big)(1) = 3, \quad \textit{since } g_1(1) = 3.$$

f	0	1	2
0	0	1	2
1	0	1	1
2	0	2	0

	g_1	g_2
0	0	1
1	3	0
2	1	3

	$f(g_1, g_2)$
0	1
1	3
2	3

Figure 2.2: The composition of a binary and two unary partial operations on E_3.

Definition 2.2.2 *Set $C \subseteq P_k$ is called* partial clone *on E_k if the following two conditions are satisfied:*

(i) C contains all projections and

(ii) C is closed with respect to composition of partial operations.

Example 2.2.3 *(i) Every total clone is trivially a partial clone.*

(ii) Set $O_k \cup \langle c_k \rangle_p$ consists of all total operations on E_k and all nowhere defined partial operations. It is obviously a partial clone since $J_k \subset O_k \subset O_k \cup \langle c_k \rangle_p$ and it is closed with respect to composition, because composing a nowhere defined partial operation with any other (partial) operation yields again a nowhere defined partial operation.

For a set $F \subseteq P_k$ we denote by $\langle F \rangle_p$ the least partial clone containing F and it holds

$$\langle F \rangle_p = \bigcap \big\{ C \subseteq P_k : C \text{ is a partial clone and } F \subseteq C \big\}.$$

We say that $\langle F \rangle_p$ is a partial clone generated by F.

The set of all partial clones on E_k forms an algebraic lattice $\mathcal{L}_k^p$ with respect to set inclusion. Lattice operations are defined as

$$C_1 \wedge_p C_2 = C_1 \cap C_2 \qquad \text{and} \qquad C_1 \vee_p C_2 = \langle C_1 \cup C_2 \rangle_p.$$

Similarly to the case of total clones we can define Mal'tsev-type operations on P_k:

- for $f \in P_k^{(1)}$ let $\zeta f = \tau f = \Delta f = f$;
- for $f \in P_k^{(n)}, n \geq 2$, let $\zeta f, \tau f \in P_k^{(n)}$ and $\Delta f \in P_k^{(n-1)}$ be defined as

$$\begin{aligned}
&(\zeta f)(x_1, x_2, \ldots, x_n) = f(x_2, \ldots, x_n, x_1),\\
&(\tau f)(x_1, x_2, x_3, \ldots, x_n) = f(x_2, x_1, x_3, \ldots, x_n),\\
&(\Delta f)(x_1, x_2, \ldots, x_{n-1}) = f(x_1, x_1, x_2, \ldots, x_{n-1});
\end{aligned}$$

- for $f \in P_k^{(n)}$ and $g \in P_k^{(m)}$ let $f \star g \in P_k^{(m+n-1)}$ be defined as $(f \star g)(x_1, \ldots, x_{m+n-1}) =$

$$= \begin{cases} f\big(g(x_1, \ldots, x_m), x_{m+1}, \ldots, x_{m+n-1}\big), & \text{if } \ g(x_1, \ldots, x_m) \in E_k^n, \\ k, & \text{otherwise.} \end{cases} \tag{2.2}$$

It is easy to see that if $f, g \in O_k$, then $f \star g = f * g$.

The algebra $\mathcal{P}_k = (P_k; \star, \zeta, \tau, \Delta, e_1^2)$ is called the *full algebra of partial operations.*

Theorem 2.2.4 *A set $C \subseteq P_k$ is a partial clone if and only if it is a subuniverse of the algebra $\mathcal{P}_k$.*

At the end of this section we give the definition of a strong partial clone, since strong partial clones play an important role in the investigation of the lattice $\mathcal{L}_k^p$.

Definition 2.2.5 *Let $f, g \in P_k^{(n)}$. Then we say that g is a* subfunction *of f (short: $g \leq f$) if for every $\vec{x} \in E_k^n$ it holds $g(\vec{x}) \in \{f(\vec{x}), k\}$.*

A partial clone $C \subseteq P_k$ is called strong *if it contains all subfunctions of its functions, i.e., $(\forall f \in C)\,(\forall g \in P_k)\ (g \leq f \Rightarrow g \in C)$.*

2.3 Clones of incompletely specified operations

Results from this section appeared in papers [15] and [16].

A mapping from E_k^n to E_{k+1} is said to be an *incompletely specified operation* (*IS operation*, in short) on E_k, where k is considered as the unknown value. The set of all n-ary IS operations on E_k will be denoted by $I_k^{(n)}$, and the set of all IS operations by I_k, i.e., $I_k = \bigcup_{n\geq 1} I_k^{(n)}$. For $F \subseteq I_k$, let $F \cap I_k^{(n)}$ be denoted by $F^{(n)}$.

Notice that, if we disregard the interpretation of the output value k, the sets P_k and I_k are basically the same, that is, they contain the same functions. It is the way we compose those functions that makes the difference.

We will now introduce one binary relation and one binary operation on E_{k+1} that will be used in the definition of the composition of incompletely specified operations. The binary relation $\sqsubseteq$ on E_{k+1} is defined by

$$\sqsubseteq = \begin{pmatrix} 0 & 0 & 1 & 1 & \dots & k-1 & k-1 & k \\ 0 & k & 1 & k & \dots & k-1 & k & k \end{pmatrix},$$

which can be rewritten as

$$\sqsubseteq = \big\{(x,x) : x \in E_{k+1}\big\} \cup \big\{(x,k) : x \in E_k\big\}.$$

The relation $\sqsubseteq$ is obviously a partial order on E_{k+1} (Figure 2.3).

We will write $(y_1, \dots, y_n) \sqsubseteq (x_1, \dots, x_n)$ in case $y_i \sqsubseteq x_i$ for all $1 \leq i \leq n$.

Example 2.3.1 *All quadruples in relation $\sqsubseteq$ with $(1,2,0,2)$ are $(1,0,0,0)$, $(1,0,0,1)$, $(1,1,0,0)$ and $(1,1,0,1)$.*

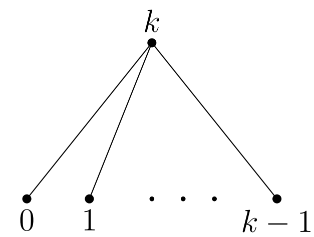

Figure 2.3: Hasse diagram of the partialy ordered set $(E_{k+1}, \sqsubseteq)$.

The binary operation $\sqcap$ on E_{k+1} is defined by

$$x_1 \sqcap x_2 = \begin{cases} x_1, & \text{if } x_1 = x_2, \\ k, & \text{otherwise.} \end{cases}$$

It is clear that the operation $\sqcap$ is commutative and associative. Thus,

$$\prod_{i=1}^{n} x_i = \begin{cases} x_1, & \text{if } x_1 = x_2 = \ldots = x_n, \\ k, & \text{otherwise.} \end{cases}$$

We can apply this operation on n-tuples by coordinates, i.e.,

$$(x_1, \ldots, x_n) \sqcap (y_1, \ldots, y_n) = (x_1 \sqcap y_1, \ldots, x_n \sqcap y_n).$$

Note that $a \sqsubseteq a \sqcap b$ for all $a, b \in E_{k+1}$ and, consequently,

$$X \subseteq Y \quad \Rightarrow \quad \prod X \sqsubseteq \prod Y.$$

Finally, we can define the composition of incompletely specified operations.

Definition 2.3.2 *Let $f \in I_k^{(n)}$ and $g_1, \ldots, g_n \in I_k^{(m)}$. The* composition *of IS operations f and $g_1, \ldots, g_n$ is the m-ary IS operation $f(g_1, \ldots, g_n)$ defined by*

$$\big(f(g_1, \ldots, g_n)\big)(\vec{x}) = \prod_{i=1}^{n} \big\{ f(\vec{y}) : \vec{y} \in E_k^n, y_i \sqsubseteq g_i(\vec{x}) \big\}, \tag{2.3}$$

where $\vec{x} = (x_1, \ldots, x_m)$ and $\vec{y} = (y_1, \ldots, y_n)$.

In more detail, for all $i \in \{1, \ldots, n\}$, if $g_i(\vec{x}) \in E_k$, then $y_i = g_i(\vec{x})$, and if $g_i(\vec{x}) = k$, then y_i takes all the values from E_k. That is, we take all the n-tuples $(y_1, \ldots, y_n)$ from E_k^n such that $(y_1, \ldots, y_n) \sqsubseteq \big(g_1(\vec{x}), \ldots, g_n(\vec{x})\big)$. Lastly, we make a "product" of the values we obtain by applying IS operation f to those n-tuples.

Contrary to the case of the composition of partial operations, although some of the IS operations $f, g_1, \ldots, g_n$ have the output k, in certain cases the output of their composition is in E_k (if all $f(\vec{y})$ have the same value from E_k), as shown in the next example.

Example 2.3.3 *In Figure 2.4 we see a binary IS operation f and two unary IS operations g_1, g_2, with their composition $f(g_1, g_2)$. For instance, since $g_1(1) = 3$, and $(0,0), (1,0), (2,0)$ are all pairs in relation $\sqsubseteq$ with $(3,0)$, we have*

$$\big(f(g_1, g_2)\big)(1) = f(0,0) \sqcap f(1,0) \sqcap f(2,0) = 0 \sqcap 0 \sqcap 0 = 0.$$

f	0	1	2
0	0	1	2
1	0	1	1
2	0	2	0

	g_1	g_2
0	0	1
1	3	0
2	1	3

	$f(g_1, g_2)$
0	1
1	0
2	3

Figure 2.4: The composition of a binary and two unary IS operations on E_3.

Note that if $f, g_1, \ldots, g_n$ are completely specified, that is, if $f, g_1, \ldots, g_n$ are total operations, composition of IS operations coincides with the composition of total operations.

The new composition naturally induces the following definition of a clone of IS operations.

Definition 2.3.4 *A set $C \subseteq I_k$ is a* clone of incompletely specified operations *(or* IS clone*) if*

(i) C contains all projections, and

(ii) C is closed under composition of IS operations.

The intersection of an arbitrary family of IS clones is an IS clone. For $F \subseteq I_k$, let $\langle F\rangle_{\text{IS}}$ denote the least IS clone that contains F,

$$\langle F\rangle_{\text{IS}} = \bigcap \big\{C \subseteq I_k : C \text{ is an IS clone and } F \subseteq C\big\}.$$

We say that the IS clone $\langle F\rangle_{\text{IS}}$ is generated by F.

The set $\mathcal{L}_k^{\text{IS}}$ of all IS clones is an algebraic lattice with respect to the set inclusion, with operations

$$C_1 \wedge_{\text{IS}} C_2 = C_1 \cap C_2 \qquad \text{and} \qquad C_1 \vee_{\text{IS}} C_2 = \langle C_1 \cup C_2\rangle_{\text{IS}}.$$

Equivalently as above, the Mal'tsev-type definition of an IS clone can be introduced as follows. Consider the following operations on I_k :

- for $f \in I_k^{(1)}$ let $\zeta f = \tau f = \Delta f = f$;
- for $f \in I_k^{(n)}, n \geq 2$, let $\zeta f, \tau f \in I_k^{(n)}$ and $\Delta f \in I_k^{(n-1)}$ be defined as

$$(\zeta f)(x_1, x_2, \ldots, x_n) = f(x_2, \ldots, x_n, x_1),$$
$$(\tau f)(x_1, x_2, x_3, \ldots, x_n) = f(x_2, x_1, x_3, \ldots, x_n),$$
$$(\Delta f)(x_1, x_2, \ldots, x_{n-1}) = f(x_1, x_1, x_2 \ldots, x_{n-1});$$

- for $f \in I_k^{(n)}$ and $g \in I_k^{(m)}$ let $f \diamond g \in I_k^{(m+n-1)}$ be defined as

$$(f \diamond g)(x_1, \ldots, x_{m+n-1}) = \prod_{\substack{y \,\in\, E_k \\ y \,\sqsubseteq\, g(x_1, \ldots, x_m)}} f(y, x_{m+1}, \ldots, x_{m+n-1}) \quad (2.4)$$

Obviously, if f and g are total operations, then $f \diamond g = f * g$.

The algebra $\mathcal{I}_k = (I_k; \diamond, \zeta, \tau, \Delta, e_1^2)$ is said to be the *full algebra of incompletely specified operations.*

Theorem 2.3.5 *A set $C \subseteq I_k$ is an IS clone if and only if it is a subuniverse of the algebra $\mathcal{I}_k$.*

We will present several additional properties of the relation $\sqsubseteq$.

Definition 2.3.6 *If n-ary IS operations f and g satisfy*

$$g(x_1, \ldots, x_n) \sqsubseteq f(x_1, \ldots, x_n)$$

for all $(x_1, \ldots, x_n) \in E_k^n$, then we say that g is an IS suboperation *of f (or f is an* IS superoperation *of g) and we write $g \sqsubseteq f$.*

Lemma 2.3.7 *Composition of IS operations is monotone with respect to the relation $\sqsubseteq$.*

Proof. Let $f, f' \in I_k^{(n)}$ and $g_1, \dots, g_n, g_1', \dots, g_n' \in I_k^{(m)}$ be IS operations such that $f' \sqsubseteq f, g_1' \sqsubseteq g_1, \dots, g_n' \sqsubseteq g_n$. For $(x_1, \dots, x_m) \in E_k^m$ denote

$$Y = \left\{(y_1, \dots, y_n) \in E_k^n : y_i \sqsubseteq g_i(x_1, \dots, x_m), 1 \leq i \leq n\right\} \quad \text{and}$$
$$Y' = \left\{(y_1, \dots, y_n) \in E_k^n : y_i \sqsubseteq g_i'(x_1, \dots, x_m), 1 \leq i \leq n\right\}.$$

Since $g_i'(x_1, \dots, x_m) \sqsubseteq g_i(x_1, \dots, x_m)$, for all $1 \leq i \leq n$, it is easy to see that $Y' \subseteq Y$. Therefore it holds that

$$\begin{aligned} f'(g_1', \dots, g_n')(x_1, \dots, x_m) &= \prod \left\{f'(y_1, \dots, y_n) : (y_1, \dots, y_n) \in Y'\right\} \\ &\sqsubseteq \prod \left\{f'(y_1, \dots, y_n) : (y_1, \dots, y_n) \in Y\right\} \\ &\sqsubseteq \prod \left\{f(y_1, \dots, y_n) : (y_1, \dots, y_n) \in Y\right\} \\ &= f(g_1, \dots, g_n)(x_1, \dots, x_m). \end{aligned}$$

□

It is easily observable that for $f \in I_k^{(n)} (= P_k^{(n)})$ and $g_1, \dots, g_n \in I_k^{(m)} (= P_k^{(m)})$ we have

$$f(g_1, \dots, g_n)_{\mathrm{IS}} \sqsubseteq f(g_1, \dots, g_n)_p.$$

Therefore, an IS clone is closed with respect to the composition of partial operations if it contains all IS superoperations of its elements. Moreover, in the case of partial operations IS superoperations are in fact subfunctions, and the partial clone containing all subfunctions of its elements is a strong partial clone.

2.4 Clones of hyperoperations

Let $\mathcal{P}(E_k)$ be the power set of E_k. An n-ary *hyperoperation* f on E_k is a mapping

$$f : E_k^n \to \mathcal{P}(E_k) \setminus \{\emptyset\}.$$

We will write $P_{E_k}^*$ for $\mathcal{P}(E_k) \setminus \{\emptyset\}$. Let $H_k^{(n)} = (P_{E_k}^*)^{E_k^n}$ be the set of all n-ary hyperoperations on E_k, $n \geq 1$, and $H_k = \bigcup\limits_{n \geq 1} H_k^{(n)}$ be the set of all finitary hyperoperations on E_k. For $F \subseteq H_k$ let $F^{(n)} = F \cap H_k^{(n)}$.

An n-ary hyperoperation whose values are all singletons can be considered as an n-ary operation (if we identify each singleton value $\{a\}$ with the element a). We will identify the clone of all total operations and the clone of all hyperoperations with singleton values, and they will both be denoted by the same symbol O_k.

An i-th n-ary *(hyper)projection* on E_k, $1 \leq i \leq n$, is the n-ary hyperoperation $e_i^n \in H_k^{(n)}$ defined by $e_i^n(x_1, \ldots, x_i, \ldots, x_n) = \{x_i\}$.

Let $f \in H_k^{(n)}$ and $g_1, \ldots, g_n \in H_k^{(m)}$, for positive integers m and n. The composition of hyperoperations f and $g_1, \ldots, g_n$ is the m-ary hyperoperation $f(g_1, \ldots, g_n)$ defined by

$$\big(f(g_1, \ldots, g_n)\big)(\vec{x}) = \bigcup_{i=1}^{n} \big\{ f(\vec{y}) : \vec{y} \in E_k^n, y_i \in g_i(\vec{x}) \big\}, \tag{2.5}$$

where $\vec{x} = (x_1, \ldots, x_m)$ and $\vec{y} = (y_1, \ldots, y_n)$.

Example 2.4.1 *Figure 2.5 shows us a binary hyperoperation f and two unary hyperoperations g_1, g_2, and their composition $f(g_1, g_2)$. For example, we have*

$$\big(f(g_1, g_2)\big)(1) = f(0,0) \cup f(1,0) = \{0\} \cup \{0\} = \{0\}.$$

f	0	1	2
0	$\{0\}$	$\{1\}$	$\{2\}$
1	$\{0\}$	$\{1\}$	$\{1\}$
2	$\{0\}$	$\{2\}$	$\{0\}$

	g_1	g_2
0	$\{0\}$	$\{1\}$
1	$\{0,1\}$	$\{0\}$
2	$\{1\}$	$\{0,1\}$

	$f(g_1, g_2)$
0	$\{1\}$
1	$\{0\}$
2	$\{0,1\}$

Figure 2.5: Composition of a binary and two unary hyperoperations on E_3.

Definition 2.4.2 *Set $C \subseteq H_k$ is called a* clone of hyperoperations *(or* hyperclone*) on E_k if the following two conditions are satisfied:*

(i) C contains all (hyper)projections and

(ii) C is closed with respect to composition of hyperoperations.

For a set F of hyperoperations, the least hyperclone containing F is

$$\langle F\rangle_h = \bigcap\Big\{C \subseteq H_k : C \text{ is a hyperclone and } F \subseteq C\Big\}.$$

Analogous to the case of clones, the set of all hyperclones, ordered by set inclusion, and denoted by $\mathcal{L}_k^h$, is an algebraic lattice. Lattice operations are defined by

$$C_1 \wedge_h C_2 = C_1 \cap C_2 \qquad \text{and} \qquad C_1 \vee_h C_2 = \langle C_1 \cup C_2\rangle_h.$$

Here we introduce the full algebra of hyperclones, with the following operations on H_k :

- for $f \in H_k^{(1)}$ let $\zeta f = \tau f = \Delta f = f$;
- for $f \in H_k^{(n)}, n \geq 2$, let $\zeta f, \tau f \in H_k^{(n)}$ and $\Delta f \in H_k^{(n-1)}$ be defined as

$$\begin{aligned}(\zeta f)(x_1, x_2, \ldots, x_n) &= f(x_2, \ldots, x_n, x_1),\\ (\tau f)(x_1, x_2, x_3, \ldots, x_n) &= f(x_2, x_1, x_3, \ldots, x_n),\\ (\Delta f)(x_1, x_2, \ldots, x_{n-1}) &= f(x_1, x_1, x_2, \ldots, x_{n-1});\end{aligned}$$

- for $f \in H_k^{(n)}$ and $g \in H_k^{(m)}$ let $f \circ g \in H_k^{(m+n-1)}$ be defined as

$$(f \circ g)(x_1, \ldots, x_{m+n-1}) = \bigcup_{y \in g(x_1,\ldots,x_m)} f(y, x_{m+1}, \ldots, x_{m+n-1}). \tag{2.6}$$

Clearly, if $f, g \in O_k$, then $f \circ g = f * g$.

The algebra $\mathcal{H}_k = (H_k; \circ, \zeta, \tau, \Delta, e_1^2)$ is called the *full algebra of hyperoperations.*

We can prove the following theorem similarly as in the case of clones.

Theorem 2.4.3 *A set $C \subseteq H_k$ is a hyperclone if and only if it is a subuniverse of the algebra $\mathcal{H}_k$.*

Now we introduce a relation on the set of all hyperoperations, which will be used later in order to define two special classes of hyperclones.

Definition 2.4.4 *If $f, g \subseteq H_k^{(n)}$ satisfy*

$$g(x_1, \ldots, x_n) \subseteq f(x_1, \ldots, x_n) \text{ for all } (x_1, \ldots, x_n) \in E_k^n$$

then g is said to be a sub-hyperoperation *of f (or f is said to be a* super-hyperoperation *of g). We write $g \subseteq f$.*

It is easy to prove that the composition of hyperoperations is monotone with respect to inclusion, i.e., if $g_1' \subseteq g_1, \ldots, g_n' \subseteq g_n$ and $f' \subseteq f$ then $f'(g_1', \ldots, g_n') \subseteq f(g_1, \ldots, g_n)$.

Concluding remark

It is obvious that both partial composition and IS composition applied to total operations coincide with the total composition. Therefore the lattice $\mathcal{L}_k$ of total clones is a sublattice of both the lattice $\mathcal{L}_k^p$ of partial clones and the lattice $\mathcal{L}_k^{\mathrm{IS}}$ of IS clones.

In the case of hyperclones, we can assign to every total operation f a hyperoperation f^h such that $f^h(\vec{x}) = \{f(\vec{x})\}$, $\vec{x} \in E_k^n$, and this mapping induces a full order embedding of the lattice $\mathcal{L}_k$ into the lattice $\mathcal{L}_k^h$ of hyperclones.

The subsequent chapter provides us with the opposite direction, that is, embeddings of the lattices $\mathcal{L}_k^p$, $\mathcal{L}_k^{\mathrm{IS}}$ and $\mathcal{L}_k^h$ into the corresponding lattices of total clones.

Chapter 3

Extensions

In this chapter we will demonstrate a method of assigning to every partial operation, incompletely specified operation and hyperoperation a suitable total operation and investigate thereby induced embeddings of lattices of partial operations, incompletely specified operations and hyperoperations into a corresponding lattice of total operations.

3.1 One-point extension of partial operations

Let us define a map $f \mapsto f_+$ from the set P_k of all partial operations on E_k to the set O_{k+1} of all total operations on E_{k+1} by

$$f_+(\vec{x}) = \begin{cases} f(\vec{x}), & \text{if } \vec{x} \in E_k^n, \\ k, & \text{otherwise,} \end{cases}$$

for all $n > 0$ and $\vec{x} \in E_{k+1}^n$. We call f_+ the *extended partial operation* (or the *one-point extension*) of f. For $F \subseteq P_k$, we put

$$F_+ = \{f_+ : f \in F\}.$$

One-point extensions of Boolean functions `AND` and `OR` are presented in Figure 3.1.

Clearly, for $f \in P_k^n$ and $g, g_1, \ldots, g_n \in P_k^m$ we can write

$$\bigl(f(g_1, \ldots, g_n)\bigr)(x_1, \ldots, x_m) = f_+\bigl(g_1(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m)\bigr) \text{ and}$$

$$(f \star g)(x_1, \ldots, x_{m+n-1}) = f_+\bigl(g(x_1, \ldots, x_m), x_{m+1}, \ldots, x_{m+n-1}\bigr)$$

$\mathtt{AND}_+$	0	1	2
0	0	0	2
1	0	1	2
2	2	2	2

$\mathtt{OR}_+$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

Figure 3.1: Operations $\mathtt{AND}_+$ and $\mathtt{OR}_+$.

instead of (2.1) and (2.2).

As an immediate consequence of the definition of this extension we get

(1) for $f \in P_k^{(n)}$ and $g_1, \dots, g_n \in P_k^{(m)}$ it holds

$$\big(f(g_1, \dots, g_n)\big)_+ = f_+\big((g_1)_+, \dots, (g_n)_+\big);$$

(2) for $f \in P_k^{(n)}$ and $g \in P_k^{(m)}$ it holds $\alpha(f_+) = (\alpha f)_+$, for every operation $\alpha \in \{\zeta, \tau, \Delta\}$, and also $f_+ \star g_+ = (f \star g)_+$.

However, extension of a partial clone on E_k is not a clone on E_{k+1}, since it does not contain projections. Although we would expect that the projection $e_i^{n,E_{k+1}}$ should be extension of e_i^{n,E_k}, this is not the case, as it is shown in Figure 3.2.

e_1^{2,E_3}	0	1	2
0	0	0	0
1	1	1	1
2	2	2	2

$(e_1^{2,E_2})_+$	0	1	2
0	0	0	2
1	1	1	2
2	2	2	2

Figure 3.2: First binary projection on E_3 and the one-point extension of first binary projection on E_2.

Moreover, there is no n-ary partial operation f on E_k such that $e_i^{n,E_{k+1}} = f_+$, since for every n-tuple $(x_1, \dots, x_n) \in E_{k+1}^n$ with $x_i \neq k$ and some $x_j = k$ we have

$$e_i^{n,E_{k+1}}(x_1, \dots, x_n) = x_i \neq k = f_+(x_1, \dots, x_n).$$

Reversely, we can define a map $g \mapsto g_-$ from O_{k+1} to P_k in the following way

$$g_-(\vec{x}) = g(\vec{x}), \ \ \vec{x} \in E_k^n.$$

We call g_- the *restricted function* of g, and for $G \subseteq O_{k+1}$ we denote

$$G_- = \{g_- : g \in G\}.$$

It is obvious that for every partial operation $f \in P_k$ we have $(f_+)_- = f$. However, if $g \in O_{k+1}$ is not the extension of some partial operation on E_k, then $(g_-)_+ \neq g$, as it is shown in the following example.

Example 3.1.1 *We can see in Figure 3.3 that g is an operation on E_3 which is not the extension of any partial operation on E_2, and therefore $(g_-)_+ \neq g$.*

$$\begin{array}{c|ccc} g & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 1 \\ 1 & 2 & 1 & 0 \\ 2 & 2 & 2 & 1 \end{array} \quad \to \quad \begin{array}{c|cc} g_- & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 2 & 1 \end{array} \quad \to \quad \begin{array}{c|ccc} (g_-)_+ & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 2 \\ 1 & 2 & 1 & 2 \\ 2 & 2 & 2 & 2 \end{array}$$

Figure 3.3: Extension of a restricted operation.

The subsequent theorem states that the lattice of partial clones on E_k is isomorphic to the sublattice of the lattice of clones on E_{k+1}.

Theorem 3.1.2 ([58, 6, 7])

(1) For every partial clone $F \subseteq P_k$ we have $F = (\langle F_+ \rangle)_-$.

(2) For every clone $G \subseteq O_{k+1}$, with $\langle (J_k)_+ \rangle \subseteq G$, the set G_- is a partial clone on E_k with the property $G = (\langle G_- \rangle_p)_+$.

(3) The mapping

$$\varphi : \mathcal{L}(\langle (J_k)_+ \rangle; \langle (H_k)_+ \rangle) \to \mathcal{L}_k^p, \ \text{given by } G \mapsto G_-,$$

is a lattice isomorphism between the lattices $\mathcal{L}(\langle (J_k)_+ \rangle; \langle (H_k)_+ \rangle)$ and $\mathcal{L}_k^p$, where $\varphi^{-1}(F) = \langle F_+ \rangle$ holds for every $F \in \mathcal{L}_k^p$.

3.2 One-point extension of IS operations

Next we extend mappings from I_k to operations on the set E_{k+1} and introduce a corresponding definition of an extended IS clone. Results of this section are published in [16].

Let us define a map $f \mapsto f^+$ from the set I_k of all IS operations on E_k to the set O_{k+1} of all total operations on E_{k+1} by

$$f^+(\vec{x}) = \prod \left\{ f(\vec{y}) : \vec{y} \in E_k^n \text{ and } \vec{y} \sqsubseteq \vec{x} \right\},$$

for all $n > 0$ and $\vec{x} \in E_{k+1}^n$. We call f^+ the *extended IS operation* (or the *one-point extension*) of f. For $F \subseteq I_k$, the set of all extended IS operations from F will be denoted by F^+. Note that the map $f \mapsto f^+$ is injective.

In more detail,

$$f^+(x_1, \ldots, x_n) = \begin{cases} f(x_1, \ldots, x_n), & \text{if } (x_1, \ldots, x_n) \in E_k^n, \\ a, & \text{if case } (*) \text{ holds} \\ k, & \text{otherwise,} \end{cases} .$$

Case $(*)$ happens if f maps all n-tuples, with coordinates from E_k, and which are in relation with $(x_1, \ldots, x_n) \in E_{k+1}^n$, to the same value $a \in E_k$.

One-point extensions of Boolean functions `AND` and `OR` are presented in Figure 3.4.

`AND`$^+$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	2

`OR`$^+$	0	1	2
0	0	1	2
1	1	1	1
2	2	1	2

Figure 3.4: Kleene's strong ternary logic functions `AND`$^+$ and `OR`$^+$.

We can use the following notation

$$\big(f(g_1, \ldots, g_n)\big)(x_1, \ldots, x_m) = f^+\big(g_1(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m)\big) \text{ and}$$

$$(f \diamond g)(x_1, \ldots, x_{m+n-1}) = f^+\big(g(x_1, \ldots, x_m), x_{m+1}, \ldots, x_{m+n-1}\big)$$

instead of (2.3) and (2.4).

If C is an IS clone on E_k, then C^+ need not be a clone on E_{k+1}. Contrary to the case of extended partial clones, C^+ does contain all the projections, since now $J_{k+1} = J_k^+$, but it is not closed with respect to composition, meaning that the composition of extended IS operations is not necessary an extended IS operation, which we illustrate by the following example.

Example 3.2.1 *In Figure 3.5 we have extended IS operations* $\mathtt{OR}^+$, g_1^+, g_2^+ *and their composition* $h = \mathtt{OR}^+(g_1^+, g_2^+)$. *Suppose that there exists* $f \in I_2$ *such that* $f^+ = h$. *Then* $f(0) = f(1) = 1$, *but*

$$f^+(2) = f(0) \sqcap f(1) = 1 \sqcap 1 = 1 \neq 2 = h(2).$$

Thus, h is not an extended IS operation.

$\mathtt{OR}^+$	0	1	2
0	0	1	2
1	1	1	1
2	2	1	2

	g_1^+	g_2^+
0	1	2
1	2	1
2	2	2

	h
0	1
1	1
2	2

Figure 3.5: The composition of extended IS operations

Also if we consider Mal'tsev operations, set of all extended IS operations I_k^+ is not closed under Δ, for $k \geq 2$, and it is not closed under $*$, for $k \geq 3$.

Lemma 3.2.2 **([16])**

(a) *For $k \geq 2$ there exists $f \in I_k$ such that $(\Delta f)^+ \neq \Delta(f^+)$.*

(b) *For $f \in I_2^{(n)}$ and $g \in I_2^{(m)}$ it always holds that $f^+ * g^+ = (f \diamond g)^+$.*

(c) *For $k \geq 3$, there exist $f, g \in I_k$ such that $f^+ * g^+ \neq (f \diamond g)^+$.*

Proof.

(a) Let $f \in I_k$, for $k \geq 2$, satisfy $f(0,1) = 1$ and $f(i,i) = 0$ for every $i \in E_k$. Then

$$\begin{aligned}(\Delta f)^+(k) &= \Delta f(0) \sqcap \Delta f(1) \sqcap \ldots \sqcap \Delta f(k-1)\\ &= f(0,0) \sqcap f(1,1) \sqcap \ldots \sqcap f(k-1,k-1) = 0 \text{ and}\end{aligned}$$

$$\begin{aligned}\Delta f^+(k) &= f^+(k,k) = f(0,0) \sqcap f(0,1) \sqcap \cdots \sqcap f(k-1,k-1)\\ &= 0 \sqcap 1 \sqcap \cdots \sqcap 0 = k.\end{aligned}$$

(b) It follows from the fact that in the two element case we have

$$f^+(y_1 \sqcap y_2, x_2, \ldots, x_n) = f^+(y_1, x_2, \ldots, x_n) \sqcap f^+(y_2, x_2, \ldots, x_n).$$

(One can notice that, in general, this does not hold for $k \geq 3$.)

(c) Let us choose $f \in I_3^{(2)}$ and $g \in I_3^{(1)}$ such that $f(0,2) = f(1,2) = 0$, $f(2,2) = 1$, $g(0) = g(1) = 0$, $g(2) = 1$. Then

$$\begin{aligned}(f \diamond g)^+(3,2) &= (f \diamond g)(0,2) \sqcap (f \diamond g)(1,2) \sqcap (f \diamond g)(2,2)\\ &= f\big(g(0),2\big) \sqcap f\big(g(1),2\big) \sqcap f\big(g(2),2\big)\\ &= f(0,2) \sqcap f(1,2) = 0 \quad \text{and}\end{aligned}$$

$$\begin{aligned}(f^+ * g^+)(3,2) &= f^+\big(g^+(3),2\big) = f^+\big(g(0) \sqcap g(1) \sqcap g(2),2\big)\\ &= f^+(3,2) = f(0,2) \sqcap f(1,2) \sqcap f(2,2) = 3,\end{aligned}$$

hence $(f \diamond g)^+ \neq (f^+ * g^+)$.

□

In order to get an algebra of extended IS operations that is isomorphic to the full algebra of IS operations, we introduce the following operations on I_k^+ :

$$\Delta_i : (I_k^+)^{(n)} \to (I_k^+)^{(n-1)}, \quad f^+ \mapsto (\Delta f)^+ \text{ and}$$

$$*_i : (I_k^+)^{(n)} \times (I_k^+)^{(m)} \to (I_k^+)^{(m)}, \quad (f^+, g^+) \mapsto (f \diamond g)^+.$$

Since the map $f \mapsto f^+$ is injective as noted above, f (g) is determined uniquely from f^+ (g^+), and therefore the operations Δ_i and $*_i$ are well-defined.

Theorem 3.2.3 ([16]) *The mapping $f \mapsto f^+$ is an isomorphism from algebra $\mathcal{I}_k = (I_k;\, \diamond,\, \zeta,\, \tau,\, \Delta,\, e_1^{2,E_k})$ to $\mathcal{I}_k^+ = (I_k^+;\, *_i,\, \zeta,\, \tau,\, \Delta_i,\, e_1^{2,E_{k+1}})$.*

Proof. It follows directly from the definitions of Δ_i and $*_i$, and from the fact that $(\zeta f)^+ = \zeta f^+$, $(\tau f)^+ = \tau f^+$ and $(\pi_1^{2,E_k})^+ = \pi_1^{2,E_{k+1}}$. We give here a proof for τ and the proof for ζ is similar.

For $\vec{x} = (x_1, x_2, \ldots, x_n)$ and $\vec{y} = (y_1, y_2, \ldots, y_n)$ we have

$$\begin{aligned}(\tau f)^+(\vec{x}) &= \prod \big\{(\tau f)(\vec{y}) \,:\, \vec{y} \in E_k^n,\, \vec{y} \sqsubseteq \vec{x}\big\}\\ &= \prod \big\{f(y_2, y_1, \ldots, y_n) \,:\, \vec{y} \in E_k^n,\, \vec{y} \sqsubseteq \vec{x}\big\}\\ &= f^+(x_2, x_1, \ldots, x_n) = \tau f^+(\vec{x}).\end{aligned}$$

□

As an immediate consequence of the previous theorem we obtain the following corollary.

Corollary 3.2.4 ([16]) *A set $C \subseteq I_k$ is an IS clone iff C^+ is a subuniverse of $\mathcal{I}_k^+$.*

3.3 Power set extension of hyperoperations

To an arbitrary n-ary hyperoperation f on E_k we can assign an n-ary operation $f^\#$ on $P^*_{E_k}$ defined as

$$f^\#(X_1,\ldots,X_n) = \bigcup \big\{ f(x_1,\ldots,x_n) \,:\, x_i \in X_i, 1 \leq i \leq n \big\},$$

for $X_1,\ldots,X_n \in P^*_{E_k}$.

We call $f^\#$ the *extended operation* of f. For an arbitrary set F of hyperoperations, let $F^\# = \{f^\# : f \in F\}$.

Here we will state a few obvious properties of this extension. For more detail we refer the reader to [59], [60] and [23].

Notice that we can write

$$\big(f(g_1,\ldots,g_n)\big)(x_1,\ldots,x_m) = f^\#\big(g_1(x_1,\ldots,x_m),\ldots,g_n(x_1,\ldots,x_m)\big) \text{ and}$$

$$(f \circ g)(x_1,\ldots,x_{m+n-1}) = f^\#\big(g(x_1,\ldots,x_m),x_{m+1},\ldots,x_{m+n-1}\big)$$

instead of (2.5) and (2.6).

Unfortunately, same as in the two previous cases, assigning the set $C^\#$ to some hyperclone C is not the desired embedding from the lattice of hyperclones on E_k into the lattice of clones on $P^*_{E_k}$, since C being a hyperclone on E_k does not imply that $C^\#$ is a clone on $P^*_{E_k}$. More precisely, it is obvious that $(e_i^{n,E_k})^\# = e_i^{n,P^*_{E_k}}$, and therefore $C^\#$ contains all projections, but composition of hyperoperations is not compatible with the operator $\#$. Namely, we have

$$\big(f(g_1,\ldots,g_n)\big)^\#(X_1,\ldots,X_n) \subseteq f^\#\big(g_1^\#(X_1,\ldots,X_n),\ldots,g_n^\#(X_1,\ldots,X_n)\big),$$

and generally equality does not hold, which we illustrate by the following example.

Example 3.3.1 *Let us define binary hyperoperations f, g_1 and g_2 on E_2 by*

x_1	x_2	f	g_1	g_2
0	0	$\{0\}$	$\{0\}$	$\{0\}$
0	1	$\{1\}$	$\{1\}$	$\{0,1\}$
1	0	$\{0\}$	$\{1\}$	$\{0,1\}$
1	1	$\{0\}$	$\{0,1\}$	$\{0\}$

Then we have

$$\begin{aligned}
\big(f(g_1,g_2)\big)^{\#}(\{0\},\{0,1\}) &= \big(f(g_1,g_2)\big)(0,0) \cup \big(f(g_1,g_2)\big)(0,1) \\
&= f^{\#}\big(g_1(0,0), g_2(0,0)\big) \cup f^{\#}\big(g_1(0,1), g_2(0,1)\big) \\
&= f^{\#}(\{0\},\{0\}) \cup f^{\#}(\{1\},\{0,1\}) \\
&= f(0,0) \cup f(1,0) \cup f(1,1) = \{0\},
\end{aligned}$$

$$\begin{aligned}
\big(f^{\#}(g_1^{\#},g_2^{\#})\big)(\{0\},\{0,1\}) &= f^{\#}\big(g_1^{\#}(\{0\},\{0,1\}), g_2^{\#}(\{0\},\{0,1\})\big) \\
&= f^{\#}\big(g_1(0,0) \cup g_1(0,1), g_2(0,0) \cup g_2(0,1)\big) \\
&= f^{\#}(\{0\} \cup \{1\}, \{0\} \cup \{0,1\}) = f^{\#}(\{0,1\},\{0,1\}) \\
&= f(0,0) \cup f(0,1) \cup f(1,0) \cup f(1,1) = \{0,1\}.
\end{aligned}$$

Therefore, instead of mapping a hyperclone C to the set $C^{\#}$ we will map it to the clone $\langle C^{\#}\rangle_{P_A^*}$. Nevertheless, it turns out that the least clone containing $C^{\#}$ can be obtained from $C^{\#}$ simply by place transformations.

Definition 3.3.2 *For a set F of (hyper)operations on E_k and for every mapping $\alpha : \{1,\dots,n\} \to \{1,\dots,m\}$, the* place transformation $\delta_\alpha : F^{(n)} \to F^{(m)}$, *is defined by*

$$\delta_\alpha(f)(x_1,\dots,x_m) = f(x_{\alpha(1)},\dots,x_{\alpha(n)}).$$

The δ-closure of the set F is the set

$$\delta(F) = \bigcup_{n\in\mathbb{N}} \{\delta_\alpha(f) \mid f \in F^{(n)},\, \alpha : \{1,\dots,n\} \to \{1,\dots,m\},\, m \in \mathbb{N}\}.$$

Example 3.3.3 *For the first unary projection e_1^1 (identity function) and transformation $\alpha_i^n : \{1\} \to \{1,\dots,n\}$, defined by $\alpha_i^n(1) = i$, for some $i \in \{1,\dots,n\}$, $n \in \mathbb{N}$, we obtain*

$$\big(\delta_{\alpha_i^n}(e_1^1)\big)(x_1,\dots,x_n) = e_1^1(x_{\alpha_i^n(1)}) = e_1^1(x_i) = x_i \quad\Rightarrow\quad \delta_{\alpha_i^n}(e_1^1) = e_i^n.$$

Therefore,

$$\begin{aligned}\delta(\{e_1^1\}) &= \left\{\delta_{\alpha_i^n}(e_1^1) \mid \alpha_i^n(1) = i,\, n \in \mathbb{N},\, 1 \leq i \leq n\right\} \\ &= \{e_i^n \mid n \in \mathbb{N},\, 1 \leq i \leq n\} = J_k.\end{aligned}$$

Theorem 3.3.4 ([23]) *Let C be a hyperclone on E_k. Then*

$$\langle C^{\#}\rangle_{P_A^*} = \delta(C^{\#}).$$

Proof is the same as in the case of lifted total clones (see [10]).

Consequently, we can define a mapping

$$\lambda : L_A^h \to L_{P_A^*},\;\; C \mapsto \delta(C^{\#}).$$

This mapping obviously is an order embedding, i.e.,

$$(\forall C_1, C_2 \in L_k^h)\ C_1 \subseteq C_2 \;\Rightarrow\; \lambda(C_1) \subseteq \lambda(C_2),$$

although not the full one, i.e.,

$$(\exists C_1, C_2 \in L_k^h)\ [\lambda(C_1), \lambda(C_2)] \setminus im\lambda \neq \emptyset.$$

Furthermore, it is a $\wedge$-semilattice embedding, i.e.,

$$(\forall C_1, C_2 \in L_k^h)\ \lambda(C_1 \cap C_2) = \lambda(C_1) \cap \lambda(C_2),$$

but not the lattice embedding, i.e.,

$$(\exists C_1, C_2 \in L_k^h)\ \langle \lambda(C_1) \cup \lambda(C_2)\rangle_{P_{E_k}^*} \subsetneq \lambda(\langle C_1 \cup C_2\rangle_h).$$

Proofs and counterexamples for previous assertions can be found in [23].

On the other hand, this extension is compatible with Mal'tsev operations $\circ, \zeta$ and τ, but the set of all extended hyperoperations $H_k^{\#}$ is not closed under Δ.

Lemma 3.3.5 *For $k \geq 2$ there exists $f \in H_k$ such that $(\Delta f)^{\#} \neq \Delta(f^{\#})$.*

Proof. Let us choose $f \in H_k^{(2)}$ such that $f(0,1) = \{1\}$ and $f(i,i) = \{0\}$ for every $i \in E_k$. Then

$$\begin{aligned}(\Delta f)^{\#}(E_k) &= \Delta f(0) \cup \Delta f(1) \cup \ldots \cup \Delta f(k-1)\\ &= f(0,0) \cup f(1,1) \cup \ldots \cup f(k-1,k-1) = \{0\} \text{ and}\end{aligned}$$

$$\Delta f^{\#}(E_k) = f^{\#}(E_k, E_k) = f(0,0) \cup f(0,1) \cup \ldots \cup f(k-1,k-1) \supseteq \{0,1\}.$$

□

Let us introduce the following operation on $H_k^{\#}$:

$$\Delta_h : (H_k^{\#})^{(n)} \to (H_k^{\#})^{(n-1)}, \quad f^{\#} \mapsto (\Delta f)^{\#}$$

The operation Δ_h is well-defined because the map $f \mapsto f^{\#}$ is injective.

Theorem 3.3.6 ([59]) *The mapping $f \mapsto f^{\#}$ is an isomorphism from algebra $\mathcal{H}_k = (H_k;\, \circ,\, \zeta,\, \tau,\, \Delta,\, e_1^{2,E_k})$ to $\mathcal{H}_k^{\#} = (H_k^{\#};\, *,\, \zeta,\, \tau,\, \Delta_h,\, e_1^{2,P^*_{E_k}})$.*

Chapter 4

Polymorphisms

Generally, it is quite difficult to explicitly describe all elements of a clone. Therefore we need a method to represent them using somewhat simpler structures. Luckily such a characterization exists and it is obtained by means of a Galois connection. Namely we have a correspondence between clones and their relational counterparts, co-clones. Co-clones are usually described using elementary operations on relations, i.e., as subuniverses of a certain algebra, but we will not go into detail about that, since we are more interested in representing clones using relations.

In this chapter definition of a Galois connection will be given, as well as the method of creating one for arbitrary sets. We proceed by describing a property of preserving a relation by an operation (resp. partial operation, IS operation, hyperoperation), which will yield corresponding Galois connection between relations and operations (resp. partial operations, IS operations, hyperoperations).

4.1 Galois connection

Let us recall what is a Galois connection in general.

Definition 4.1.1 Galois connection *between sets A and B is a pair of mappings (α, β) where*

$$\alpha : \mathcal{P}(A) \to \mathcal{P}(B) \quad \textit{and} \quad \beta : \mathcal{P}(B) \to \mathcal{P}(A),$$

such that for all $X, X_1, X_2 \in \mathcal{P}(A)$ and $Y, Y_1, Y_2 \in \mathcal{P}(B)$ it holds *(i)* $X_1 \subseteq X_2 \Rightarrow \alpha(X_1) \supseteq \alpha(X_2)$ *and* $Y_1 \subseteq Y_2 \Rightarrow \beta(Y_1) \supseteq \beta(Y_2)$;

(ii) $X \subseteq \beta(\alpha(X))$ *and* $Y \subseteq \alpha(\beta(Y))$.

It directly follows from the definition of Galois connection (α, β) between sets A and B that mappings $\alpha\beta$ and $\beta\alpha$ are closure operators on B and A, respectfully.

Next we show how to construct a Galois connection between arbitrary sets A and B.

Theorem 4.1.2 *For nonempty sets A and B, and $R \subseteq A \times B$ we define mappings*

$$\overrightarrow{R} : \mathcal{P}(A) \to \mathcal{P}(B) \quad \text{and} \quad \overleftarrow{R} : \mathcal{P}(B) \to \mathcal{P}(A)$$

by

$$\overrightarrow{R}(X) = \big\{y \in B : (\forall x \in X)\,(x, y) \in R\big\},\ X \subseteq A,$$

$$\overleftarrow{R}(Y) = \big\{x \in A : (\forall y \in Y)\,(x, y) \in R\big\},\ Y \subseteq B.$$

Then the pair $(\overrightarrow{R}, \overleftarrow{R})$ is a Galois connection between sets A and B.

This construction will be used excessively in the remainder of the current chapter.

4.2 Operations preserving relations

We will write a relation in a form of a matrix whose columns are elements of the relation.

Example 4.2.1 *Relation $\rho = \big\{(1, 0, 4, 2), (2, 3, 2, 1), (0, 1, 3, 4)\big\}$ can be written as*

$$\rho = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \\ 4 & 2 & 3 \\ 2 & 1 & 4 \end{pmatrix}.$$

Let us denote by $R_k^{(\ell)}$ the set of all ℓ-ary relations on E_k, i.e., $R_k^{(\ell)} = \mathcal{P}(E_k^\ell)$, and let $R_k = \bigcup\limits_{\ell \geq 1} R_k^{(\ell)}$ be the set of all finitary relations on E_k.

For a relation $\rho \in R_k^{(\ell)}$ denote by ρ_n^* the set of $\ell \times n$ matrices over E_k whose columns are all in ρ, and by $\rho^* = \bigcup_{n \geq 1} \rho_n^*$ the set of all such matrices over E_k. For a matrix $M = [a_{ij}]_{\ell \times n}$, $M_{1*}, M_{2*}, \ldots, M_{\ell *}$ will represent its rows, and $M_{*1}, M_{*2}, \ldots, M_{*n}$ will represent its columns.

If $M = [a_{ij}]_{\ell \times n}$ and $f \in O_k^{(n)}$, we write

$$f(M) = f\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{\ell 1} & a_{\ell 2} & \ldots & a_{\ell n} \end{pmatrix} = \begin{pmatrix} f(a_{11}, a_{12}, \ldots, a_{1n}) \\ f(a_{21}, a_{22}, \ldots, a_{2n}) \\ \vdots \\ f(a_{\ell 1}, a_{\ell 2}, \ldots, a_{\ell n}) \end{pmatrix}.$$

Definition 4.2.2 *We say that an operation $f \in O_k^{(n)}$* preserves *a relation $\rho \in R_k^{(\ell)}$ (or ρ is* invariant *of f) if for every $\ell \times n$ matrix $M \in \rho^*$ it holds $f(M) \in \rho$.*

Example 4.2.3 *Let ρ be a ternary relation on E_2 given by*

$$\rho = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

Let $f \in E_2^{(1)}$ be a negation function and $g \in E_2^{(2)}$ logical disjunction, that is,

	f
0	1
1	0

g	0	1
0	0	1
1	1	1

Obviously, operation f preserves relation ρ since

$$f\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \in \rho, \qquad f\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \in \rho, ,$$

$$f\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in \rho, \qquad f\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \in \rho.$$

On the other hand, ρ is not invariant of g, e.g.,

$$g\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \notin \rho.$$

The set of all operations preserving a relation ρ is usually denoted by $Pol\,\rho$, and every $f \in Pol\,\rho$ is called *polymorphism* of ρ. Also the set of all relations which are preserved by operation f is denoted by $Inv\,f$.

Example 4.2.4

(1) For a relation $\delta^2_{k,\{1,2\}} = \{(x,x) : x \in E_k\}$ we have $Pol\,\delta^2_{k,\{1,2\}} = O_k$, i.e., binary diagonal relation is invariant of every operation on E_k, since for arbitrary $f \in O_k^{(n)}$ we have

$$f\begin{pmatrix} x_1 & \cdots & x_n \\ x_1 & \cdots & x_n \end{pmatrix} = \begin{pmatrix} f(x_1,\ldots,x_n) \\ f(x_1,\ldots,x_n) \end{pmatrix} \in \delta^2_{k,\{1,2\}}.$$

(2) For an arbitrary relation $\rho \in R_k$ it holds $J_k \subseteq Pol\,\rho$, i.e., every projection preserves all relations. If $\rho \subseteq E_k^\ell$, $M \in \rho^*$ and e_i^n is i-th n-ary projection, it holds

$$e_i^n(M) = \begin{pmatrix} e_i^n(x_{11},\ldots,x_{1i},\ldots,x_{1n}) \\ e_i^n(x_{21},\ldots,x_{2i},\ldots,x_{2n}) \\ \vdots \\ e_i^n(x_{\ell 1},\ldots,x_{\ell i},\ldots,x_{\ell n}) \end{pmatrix} = \begin{pmatrix} x_{1i} \\ x_{2i} \\ \vdots \\ x_{\ell i} \end{pmatrix} \in \rho.$$

Now we can define mappings

$$Pol : \mathcal{P}(R_k) \to \mathcal{P}(O_k) \quad \text{and} \quad Inv : \mathcal{P}(O_k) \to \mathcal{P}(R_k)$$

by

$$Pol\,Q = \bigcap_{\rho \in Q} Pol\,\rho = \{f \in O_k : f \text{ preserves every } \rho \in Q\},\ Q \subseteq R_k,$$

$$Inv\,F = \bigcap_{f \in F} Inv\,f = \{\rho \in R_k : \text{every } f \in F \text{ preserves } \rho\},\ F \subseteq O_k.$$

Using Theorem 4.1.2 we deduce that the pair (Pol, Inv) is a Galois connection between operations and relations.

It is easy to see that $Pol\,Q$ is a clone, for every $Q \subseteq R_k$, and also $Inv\,F$ is a co-clone for every $F \subseteq O_k$. Nevertheless, the opposite also holds, i.e., every clone is a set of polymorphisms of some set of relations and every co-clone is of the form $Inv\,F$ for some set of operations F.

Theorem 4.2.5 **([4, 27])**

(1) If $C \subseteq O_k$ is a clone, then $C = Pol(Inv\,C)$.

(2) If $Q \subseteq R_k$ is a co-clone, then $Q = Inv(Pol\,Q)$.

What is of great assistance in the investigation of the clone lattice is the fact that the bigger the clone is the smaller the corresponding set of relations is, which is the obvious conclusion from the definition of Galois connection. Specially, as we will see in Chapter 6 (Theorem 6.1.2), every maximal clone is the set of polymorphisms of a single relation.

4.3 Partial operations preserving relations

In the case of partial operations we are considering relations $\rho \subseteq E_{k+1}^{\ell}$. Again we will denote by ρ^* the set of all matrices whose columns are the elements of ρ.

Definition 4.3.1 *We say that a partial operation $f \in P_k^{(n)}$* preserves *an ℓ-ary relation ρ on E_{k+1} (or ρ is* invariant *of f) if for all $\ell \times n$ matrices $M \in \rho^*$ it holds $f_+(M) \in \rho$.*

We denote by $pPol\,\rho$ the set of all functions from P_k that preserve the relation ρ and by $pInv\,f$ the set of all relations that are preserved by f.

Furthermore, we put

$$pPOL\,\rho = pPol\big(\rho \cup (E_{k+1}^{\ell} \setminus E_k^{\ell})\big),$$

where the relation $\rho \cup (E_{k+1}^{\ell} \setminus E_k^{\ell})$ is the *full extension* of ρ.

Lemma 4.3.2 *For every $\rho \in R_{k+1}^{(\ell)}$ sets $pPol\,\rho$ and $pPOL\,\rho$ are partial clones. Moreover, $pPOL\,\rho$ is a strong partial clone.*

Proof. We will prove that $pPol\,\rho$ is a partial clone. Proof for $pPOL\,\rho$ is analogue. Same as in the case of total operations, any projection preserves every relation, hence $J_k \subseteq pPol\,\rho$. Also if $f, g_1, \dots, g_n \in pPol\rho$, then $f(g_1, \dots, g_n) \in pPol\,\rho$ since $\big(f(g_1, \dots, g_n)\big)_+ = f_+\big((g_1)_+, \dots, (g_n)_+\big)$.

Next we show that the partial clone $pPOL\,\rho$ contains all subfunctions of its elements. Let $f \in pPOL\,\rho$, $g \leq f$ and $M \in \rho^*$. If $f_+(M) = (a_1, \dots, a_\ell)$, then $g_+(M) = (b_1, \dots, b_\ell)$, where $b_i \in \{a_i, k\}$, for $i \in \{1, \dots, \ell\}$. Thus $g_+(M) \in \rho \cup (E_{k+1}^\ell \setminus E_k^\ell)$, which means that $g \in pPOL\,\rho$. □

Let us define the mappings

$$pPol : \mathcal{P}(R_{k+1}) \to \mathcal{P}(P_k) \quad \text{and} \quad pInv : \mathcal{P}(P_k) \to \mathcal{P}(R_{k+1})$$

by

$$pPol\,Q = \bigcap_{\rho \in Q} pPol\,\rho = \{f \in P_k : f \text{ preserves every } \rho \in Q\},\ Q \subseteq R_{k+1},$$

$$pInv\,F = \bigcap_{f \in F} pInv\,f = \{\rho \in R_{k+1} : \text{every } f \in F \text{ preserves } \rho\},\ F \subseteq P_k.$$

Evidently, the pair $(pPol, pInv)$ is a Galois connection between relations and partial operations.

More details about this Galois connection can be found in [68].

4.4 Hyperoperations preserving relations

There are several ways to define the property of preserving a relation by a hyperoperation. In this section we will introduce two Galois connections between hyperoperations and relations on a finite set.

4.4.1 Down closed hyperclones

The first Galois connection that will be presented here was independently studied by Börner (in [8]) and Romov (in [53], [54]). Corresponding Galois closed sets are hyperclones that contain all sub-hyperoperations of their elements, and they are called down (or restriction) closed hyperclones.

Let $F \subseteq H_k$, and let us denote by $\lfloor F \rfloor_h$ the set of all sub-hyperoperations of all the elements of F, i.e.,

$$\lfloor F \rfloor_h = \big\{ g \in H_k : (\exists f \in F)\ g \subseteq f \big\}.$$

Lemma 4.4.1 *If $C \subseteq H_k$ is a hyperclone, then $\lfloor C \rfloor_h$ is also a hyperclone.*

Proof. Since obviously $C \subseteq \lfloor C \rfloor_h$ and hyperclone C contains all projections, the same holds for $\lfloor C \rfloor_h$.

For hyperoperations $f, g_1, \dots, g_n \in \lfloor C \rfloor_h$, there exist $f', g_1', \dots, g_n' \in C$ such that $f \subseteq f'$, $g_1 \subseteq g_1'$, $\dots$, $g_n \subseteq g_n'$. Using the fact that C is a hyperclone and that the composition of hyperoperations is monotone with respect to inclusion we obtain $f'(g_1', \dots, g_n') \in C$ and $f(g_1, \dots, g_n) \subseteq f'(g_1', \dots, g_n')$ which implies $f(g_1, \dots, g_n) \in \lfloor C \rfloor_h$. □

Let us define a mapping $d : \mathcal{P}(H_k) \to \mathcal{P}(H_k)$ by $d(F) = \lfloor \langle F \rangle_h \rfloor_h$.

Lemma 4.4.2 *The mapping d is an algebraic closure operator.*

Proof.

(i) Trivially, $F \subseteq \langle F \rangle_h \subseteq \lfloor \langle F \rangle_h \rfloor_h$, i.e., $F \subseteq d(F)$.

(ii) If $F \subseteq G$, then obviously $\langle F \rangle_h \subseteq \langle G \rangle_h$. For $f \in \lfloor \langle F \rangle_h \rfloor_h$ there is $g \in \langle F \rangle_h$ such that $f \subseteq g$. However, g is also in $\langle G \rangle_h$, which implies $f \in \lfloor \langle G \rangle_h \rfloor_h$, i.e., $d(F) \subseteq d(G)$.

(iii) In order to prove $d\big(d(F)\big) = d(F)$ it is sufficient to show that $d\big(d(F)\big) \subseteq d(F)$ since the opposite inclusion holds by (i). For any $f \in d\big(d(F)\big)$ there exists $g \in \langle d(F) \rangle_h (= d(F))$ such that $f \subseteq g$. Next, there is $h \in \langle F \rangle_h$ with $g \subseteq h$. Therefore, since $h \in \langle F \rangle_h$ and $f \subseteq h$, we obtain $f \in d(F)$.

(iv) We are going to show that $d(F) = \bigcup \big\{ d(G) : G \subseteq F \text{ and } G \text{ is finite} \big\}$.

($\subseteq$) If $f \in d(F)$, there is $g \in \langle F \rangle_h$ such that $f \subseteq g$. If we choose $G = \{g\}$, then $f \in d(G)$.

($\supseteq$) Assume that there is a finite subset G of F such that $f \in d(G)$. It means that there is $g \in \langle G \rangle_h$ such that $f \subseteq g$. Now, since $\langle G \rangle_h \subseteq \langle F \rangle_h$, it follows that $g \in \langle F \rangle_h$ and $f \in d(F)$.

□

Definition 4.4.3 *A set $F \subseteq H_k$ is called* down closed hyperclone *if*

$$d(F) = F.$$

Every clone is trivially a down closed hyperclone, and obviously H_k is the largest down closed hyperclone on E_k.

Lemma 4.4.4 *If $C \subseteq H_k$ is a hyperclone, then $\lfloor C \rfloor_h$ is a down closed hyperclone.*

Proof. It follows directly from Lemma 4.4.1 and Lemma 4.4.2 (iii). □

It is also easily deduced that the intersection of an arbitrary family of down closed hyperclones is again a down closed hyperclone.

By $L_k^{h,\Downarrow}$ we will denote the set of all down closed hyperclones on E_k and $\mathcal{L}_k^{h,\Downarrow}$ will be the corresponding poset $(L_k^{h,\Downarrow}, \subseteq)$.

The following theorem is also an immediate corollary of the previous lemmas.

Theorem 4.4.5 *Down closed hyperclones form an algebraic lattice $\mathcal{L}_k^{h,\Downarrow}$ with respect to the set inclusion. The lattice operations on $\mathcal{L}_k^{h,\Downarrow}$ are defined as follows*

$$C_1 \wedge_{h,\Downarrow} C_2 = C_1 \cap C_2 \qquad \text{and} \qquad C_1 \vee_{h,\Downarrow} C_2 = d(C_1 \cup C_2).$$

From the definition of the join operation it is easily deduced that $\mathcal{L}_k^{h,\Downarrow}$ is not the sublattice of the lattice of all hyperclones on E_k, which we are going to illustrate by the next example.

Example 4.4.6 *Let us consider hyperclones $C_1 = d(\{f_1\})$ and $C_2 = d(\{f_2\})$ on E_3, where hyperoperations $f_1, f_2 \in H_3^{(1)}$ are defined by*

	0	1	2
f_1	$\{0\}$	$\{0,2\}$	$\{2\}$
f_2	$\{1\}$	$\{1\}$	$\{1,2\}$

We know that $\langle C_1 \cup C_2 \rangle_h$ contains hyperoperation $f_1(f_2)$:

	0	1	2
$f_1(f_2)$	$\{0,2\}$	$\{0,2\}$	$\{0,2\}$

Down closed hyperclone $d(C_1 \cup C_2)$ contains all sub-hyperoperation of its elements, and therefore it contains a sub-hyperoperation h of $f_1(f_2)$, given by $h(x) = \{2\}$, $x \in E_3$. However, h is not generated by $C_1 \cup C_2$, since no composition of elements from $C_1 \cup C_2$ yields $h(0) = \{2\}$. Thus, $C_1 \vee_{h,\Downarrow} C_2 \neq C_1 \vee_h C_2$.

Next we will introduce the Galois connection for which down closed hyperclones are Galois closed sets.

Definition 4.4.7 *Let $\ell \geq 1$ and $\rho \in R_k^{(\ell)}$. The* strong extension *of ρ is the relation ρ_d defined by*

$$\rho_d = \big\{(A_1, \ldots, A_\ell) \in (P^*_{E_k})^\ell \,:\, A_1 \times \cdots \times A_\ell \subseteq \rho\big\}.$$

This means that for $(A_1, \ldots, A_\ell)$ to be in ρ_d it is necessary that all ℓ-tuples $(a_1, \ldots, a_\ell) \in A_1 \times \cdots \times A_\ell$ are contained in ρ.

Example 4.4.8 *If ρ and θ are binary relations on E_2 given by $\rho = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $\theta = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, then the strong extensions of ρ and θ are*

$$\rho_d = \begin{pmatrix} \{0\} & \{0\} & \{0\} \\ \{0\} & \{1\} & \{0,1\} \end{pmatrix} \quad \textit{and} \quad \theta_d = \begin{pmatrix} \{1\} \\ \{1\} \end{pmatrix}.$$

Definition 4.4.9 *We say that hyperoperation $f \in H_k^{(n)}$ d-*preserves *relation $\rho \in R_k^{(\ell)}$ (or ρ is d-*invariant *of f) if for every $\ell \times n$ matrix M in ρ^* it holds $f(M) \in \rho_d$, i.e., $f(M_{1*}) \times \cdots \times f(M_{\ell *}) \subseteq \rho$.*

Example 4.4.10 *Consider relation ρ and its strong extension from Example 4.4.8. Than for a binary hyperoperation $f \in H_2$ given by*

f	0	1
0	$\{0\}$	$\{0,1\}$
1	$\{0,1\}$	$\{1\}$

we have

$$f\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \{0\} \\ \{0\} \end{pmatrix} \in \rho_d, \qquad f\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \{0\} \\ \{0,1\} \end{pmatrix} \in \rho_d,$$

$$f\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \{0\} \\ \{0,1\} \end{pmatrix} \in \rho_d, \qquad f\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \{0\} \\ \{1\} \end{pmatrix} \in \rho_d,$$

Thus, hyperoperation f d-preserves relation ρ.

Let $dPol\,\rho$ denote the set of all hyperoperations on E_k which d-preserve relation ρ, a $dInv\,f$ be the set of all relations on E_k which are d-invariant for hyperoperation f. We can now define the mappings

$$dPol : \mathcal{P}(R_k) \to \mathcal{P}(H_k) \quad \text{and} \quad dInv : \mathcal{P}(H_k) \to \mathcal{P}(R_k)$$

by

$$dPol\,Q = \bigcap_{\rho \in Q} dPol\,\rho = \{f \in H_k : f \text{ } d\text{-preserves every } \rho \in Q\},\ Q \subseteq R_k,$$

$$dInv\,F = \bigcap_{f \in F} dInv\,f = \{\rho \in R_k : \text{every } f \in F \text{ } d\text{-preserves } \rho\},\ F \subseteq H_k.$$

Clearly, the pair $(dPol, dInv)$ is a Galois connection between relations and hyperoperations.

Theorem 4.4.11 ([8])

(i) For any $Q \subseteq R_k$, $dPol\,Q$ is a down closed hyperclone.

(ii) If $C \subseteq H_k$ is a down closed hyperclone, then $C = dPol(dInv\,C)$.

4.4.2 Upward saturated hyperclones

Dually to the case of down closed hyperclones, described in Section 4.4.1, we may consider hyperclones that contain all super-hyperoperations of their elements. Here we present part of the results from [17].

For $F \subseteq H_k$, let $\lceil F \rceil_h$ denote the set of all super-hyperoperations of hyperoperations from F, i.e.,

$$\lceil F \rceil_h = \big\{f \in H_k : (\exists g \in F)\ g \subseteq f\big\}.$$

For the following two lemmas proofs are dual to those of Lemma 4.4.1 and Lemma 4.4.2.

Lemma 4.4.12 ([17]) *If $C \subseteq H_k$ is a hyperclone, then $\lceil C \rceil_h$ is a hyperclone.*

Let us define a mapping $u : \mathcal{P}(H_k) \to \mathcal{P}(H_k)$ by $u(F) = \lceil \langle F \rangle_h \rceil_h$.

Lemma 4.4.13 ([17]) *The mapping u is an algebraic closure operator.*

Definition 4.4.14 *A set $F \subseteq H_k$ is called* upward saturated hyperclone *if*

$$u(F) = F.$$

Analogous to the case of down closed hyperclones, we can conclude

Lemma 4.4.15 *If $C \subseteq H_k$ is a hyperclone, then $\lceil C \rceil_h$ is an upward saturated hyperclone.*

We will write $L_k^{h,\Uparrow}$ for the set of all upward saturated hyperclones on E_k and $\mathcal{L}_k^{h,\Uparrow}$ for the poset $(L_k^{h,\Uparrow}, \subseteq)$.

Using the previous lemmas, we can easily prove the following theorem.

Theorem 4.4.16 ([17]) *Upward saturated hyperclones form an algebraic lattice $\mathcal{L}_k^{h,\Uparrow}$ with respect to the set inclusion. The lattice operations are defined as follows*

$$C_1 \wedge_{h,\Uparrow} C_2 = C_1 \cap C_2 \qquad \textit{and} \qquad C_1 \vee_{h,\Uparrow} C_2 = u(C_1 \cup C_2).$$

We will describe one class of upward saturated hyperclones, by introducing a particular Galois connection.

Definition 4.4.17 ([61]) *Let $\ell \geq 1$ and $\rho \subseteq E_k^\ell$. The* weak extension *of ρ is the relation ρ_h defined by*

$$\rho_h = \big\{(A_1, \ldots, A_\ell) \in (P^*_{E_k})^\ell \,:\, (A_1 \times \ldots \times A_\ell) \cap \rho \neq \emptyset\big\}.$$

Thus, ρ_h consists of ℓ-tuples $(A_1, \ldots, A_\ell)$ of subsets of E_k such that for some $a_i \in A_i$, $i = 1, \ldots, \ell$, we have $(a_1, \ldots, a_\ell) \in \rho$.

Example 4.4.18 *If ρ is a binary relation on E_2 given by $\rho = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, then the weak extension of ρ is*

$$\rho_h = \begin{pmatrix} \{0\} & \{0\} & \{0\} & \{0,1\} & \{0,1\} & \{0,1\} \\ \{0\} & \{1\} & \{0,1\} & \{0\} & \{1\} & \{0,1\} \end{pmatrix}.$$

Definition 4.4.19 *Hyperoperation* $f \in H_k^{(n)}$ h-preserves *relation* $\rho \in R_k^{(\ell)}$ *(or* ρ *is* h-invariant *of* f*) if for every* $\ell \times n$ *matrix* M *in* ρ^* *it holds* $f(M) \in \rho_h$*, i.e.,* $\big(f(M_{1*}) \times \cdots \times f(M_{\ell *})\big) \cap \rho \neq \emptyset$.

Example 4.4.20 *Consider relation* ρ *and its strong extension from Example 4.4.18. Than for a binary hyperoperation* $g \in H_2$ *given by*

g	0	1
0	$\{0,1\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$

we have

$$g\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \{0,1\} \\ \{0,1\} \end{pmatrix} \in \rho_h, \qquad g\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \{0,1\} \\ \{0\} \end{pmatrix} \in \rho_h,$$

$$g\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \{0,1\} \\ \{1\} \end{pmatrix} \in \rho_h, \qquad g\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \{0,1\} \\ \{0,1\} \end{pmatrix} \in \rho_h,$$

Therefore, hyperoperation g *h-preserves relation* ρ.

Set $hPol\,\rho$ consists of all the hyperoperations that h-preserve relation ρ, and by $hInv\,f$ we denote the set of all relations that hyperoperation f h-preserves. Let us define the mappings

$$hPol : \mathcal{P}(R_k) \to \mathcal{P}(H_k) \quad \text{and} \quad hInv : \mathcal{P}(H_k) \to \mathcal{P}(R_k),$$

as follows:

$$hPol\,Q = \bigcap_{\rho \in Q} hPol\,\rho = \{f \in H_k : f \ h\text{-preserves each } \rho \in Q\},\ Q \subseteq R_k,$$

$$hInv\,F = \bigcap_{f \in F} hInv\,f = \{\rho \in R_k : \text{each } f \in F \ h\text{-preserves } \rho\},\ F \subseteq H_k.$$

It is clear that the pair $(hPol, hInv)$ is a Galois connection between relations and hyperoperations on E_k.

Lemma 4.4.21 ([17]) *Let* $\ell \geq 1$ *and* $\rho \subseteq E_k^\ell$*. Then* $hPol\,\rho$ *is an upward saturated hyperclone.*

Proof. Evidently $Pol\,\rho \subseteq hPol\,\rho$ and therefore $hPol\,\rho$ contains all projections. For $f \in (hPol\,\rho)^{(n)}$ and $g_1, \ldots, g_n \in (hPol\,\rho)^{(m)}$ let $h = f(g_1, \ldots, g_n)$. Suppose that $h \notin hPol\,\rho$, i.e., there exist an $\ell \times m$ matrix M in ρ^* such that

$$\big(h(M_{1*}), \ldots, h(M_{\ell *})\big) \notin \rho_h,$$

where $M_{1*}, \ldots, M_{\ell *}$ are the rows of M, or equivalently

$$\big(h(M_{1*}) \times \ldots \times h(M_{\ell *})\big) \cap \rho = \emptyset.$$

Then from $f \in hPol\,\rho$ we conclude that there is some $i \in \{1, \ldots, n\}$ such that

$$\big(g_i(M_{1*}), \ldots, g_i(M_{\ell *})\big) \notin \rho_h, \text{ i.e., } \big(g_i(M_{1*}) \times \ldots \times g_i(M_{\ell *})\big) \cap \rho = \emptyset.$$

Since $g_i \in hPol\rho$, we deduce that for some $j \in \{1, \ldots, m\}$ the column $M_{*j} \notin \rho$, which is not possible by the choice of matrix M. Therefore, $h \in hPol\,\rho$, and consequently $hPol\,\rho$ is a hyperclone.

We shall now prove that the hyperclone $hPol\,\rho$ is upward saturated. Let us suppose that $f \in (hPol\,\rho)^{(n)}$ and $f \subseteq g$. If we assume that $g \notin hPol\,\rho$, than there is an $\ell \times n$ matrix M in ρ^* such that

$$\big(g(M_{1*}), \ldots, g(M_{\ell *})\big) \notin \rho_h.$$

Therefore, we have

$$\big(f(M_{1*}), \ldots, f(M_{\ell *})\big) \subseteq \big(g(M_{1*}), \ldots, g(M_{\ell *})\big)$$

and

$$\big(g(M_{1*}) \times \ldots \times g(M_{\ell *})\big) \cap \rho = \emptyset,$$

which implies

$$\big(f(M_{1*}), \ldots, f(M_{\ell *})\big) \cap \rho = \emptyset,$$

and hence yields a contradiction. □

4.5 IS operations preserving relations

In this section we present two Galois connections between relations and IS operations that are analogue to those defined in sections 4.4.1 i 4.4.2, and we also investigate correspondence between relations on E_{k+1} and extended IS operations.

4.5.1 Down closed IS clones

Down closed IS clones are analogue to the down closed hyperclones that we presented in section 4.4.1.

If $F \subseteq I_k$, then $\lfloor F \rfloor_{\mathrm{IS}}$ denotes the set of all IS suboperations (Definition 2.3.6) of the IS operations from F, i.e.,

$$\lfloor F \rfloor_{\mathrm{IS}} = \big\{ f \in I_k : (\exists g \in F)\ f \sqsubseteq g \big\}.$$

We say that the set of IS operations F is *down closed* if and only if $\lfloor F \rfloor_{\mathrm{IS}} = F$. In case C is an IS clone, $\lfloor C \rfloor_{\mathrm{IS}}$ is also an IS clone. One can prove that intersection of down closed IS clones is again a down closed IS clone.

Same as in the case of down closed hyperclones the following theorem holds.

Theorem 4.5.1 *The set of all down closed IS clones forms an algebraic lattice $\mathcal{L}_k^{\mathrm{IS},\Downarrow}$ with respect to set inclusion and the lattice operations are given by*

$$C_1 \wedge_{\mathrm{IS},\Downarrow} C_2 = C_1 \cap C_2 \qquad \textit{and} \qquad C_1 \vee_{\mathrm{IS},\Downarrow} C_2 = \lfloor \langle C_1 \cup C_2 \rangle_{\mathrm{IS}} \rfloor_{\mathrm{IS}}$$

Next we define a relation on E_{k+1} which is analogue of the strong extension of the relation on E_k to the relation on $P^*_{E_k}$ (Definition 4.4.7). This enables us to introduce one type of preservation property between IS operations and relations that will yield a matching Galois connection and eventually give us one class of down closed IS clones.

Definition 4.5.2 *Let $\ell \geq 1$ and $\rho \subseteq E_k^\ell$. The strong extension of ρ is the relation ρ_s defined by*

$$\rho_s = \Big\{ (a_1, \ldots, a_\ell) \in E_{k+1}^\ell : \big(\forall (b_1, \ldots, b_\ell) \in E_k^\ell\big)\, (b_1, \ldots, b_\ell) \sqsubseteq (a_1, \ldots, a_\ell) \\ \Rightarrow\ (b_1, \ldots, b_\ell) \in \rho \Big\}.$$

Hence, ℓ-tuple $(a_1, \ldots, a_\ell) \in E_{k+1}^\ell$ is in ρ_s if all ℓ-tuples, with coordinates from E_k, and which are in relation $\sqsubseteq$ with $(a_1, \ldots, a_\ell)$, belong to ρ. In some cases relations are equal to their strong extensions.

Example 4.5.3 *If ρ and θ are binary relations on E_2 given by* $\rho = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ *and* $\theta = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, *then the strong extensions of ρ and θ are*

$$\rho_s = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix} \quad \textit{and} \quad \theta_s = \theta.$$

Definition 4.5.4 *An IS operation $f \in I_k^{(n)}$ is said to s-*preserve *relation $\rho \in R_k^{(\ell)}$ (or ρ is s-*invariant *of f) if for every $\ell \times n$ matrix $M \in \rho^*$ it holds $f(M) \in \rho_s$.*

Let $sPol\,\rho$ denote the set of all IS operations on E_k which s-preserve relation ρ, a $sInv\,f$ be the set of all relations on E_k which are s-invariant for hyperoperation f. Now we define the mappings

$$sPol : \mathcal{P}(R_k) \to \mathcal{P}(I_k) \quad \text{and} \quad sInv : \mathcal{P}(I_k) \to \mathcal{P}(R_k)$$

by

$$sPol\,Q = \bigcap_{\rho \in Q} sPol\,\rho = \{f \in I_k : f \ s\text{-preserves every } \rho \in Q\},\ Q \subseteq R_k,$$

$$sInv\,F = \bigcap_{f \in F} sInv\,f = \{\rho \in R_k : \text{every } f \in F \ s\text{-preserves } \rho\},\ F \subseteq I_k.$$

Obviously, the pair $(sPol, sInv)$ is a Galois connection between relations and IS operations.

Lemma 4.5.5 *For every $\rho \subseteq E_k^\ell$ set $sPol\,\rho$ is a down closed IS clone.*

4.5.2 Upward saturated IS clones

Upward saturated IS clones are dual to down closed IS clones and analogue to the upward saturated hyperclones that were studied in Section 4.4.2.

If $F \subseteq I_k$, then $\lceil F \rceil_{\mathrm{IS}}$ denotes the set of all IS superoperations (Definition 2.3.6) of the IS operations from F, i.e.,

$$\lceil F \rceil_{\mathrm{IS}} = \big\{ f \in I_k : (\exists g \in F)\ g \sqsubseteq f \big\}.$$

The set F of IS operations is said to be *upward saturated* if and only if $\lceil F\rceil_{\mathrm{IS}} = F$. Evidently, if C is an IS clone, then $\lceil C\rceil_{\mathrm{IS}}$ is also an IS clone, and moreover, intersection of an arbitrary family of upward saturated IS clones is an upward saturated IS clone.

Theorem 4.5.6 *The set of all upward saturated IS clones forms an algebraic lattice $\mathcal{L}_k^{\mathrm{IS},\Uparrow}$ with respect to set inclusion and the lattice operations are given by*

$$C_1 \wedge_{\mathrm{IS},\Uparrow} C_2 = C_1 \cap C_2 \qquad \textit{and} \qquad C_1 \vee_{\mathrm{IS},\Uparrow} C_2 = \lceil \langle C_1 \cup C_2\rangle_{\mathrm{IS}}\rceil_{\mathrm{IS}}$$

In what follows we introduce one class of upward saturated IS clones. The next definition of the relation on E_{k+1}, which is the extension of some relation on E_k, is analogue to Definition 4.4.17 in the case of hyperclones.

Definition 4.5.7 *Let $\ell \geq 1$ and $\rho \subseteq E_k^\ell$. The weak extension of ρ is the relation ρ_w defined by*

$$\rho_w = \Big\{(a_1,\dots,a_\ell) \in E_{k+1}^\ell : \big(\exists (b_1,\dots,b_\ell) \in \rho\big)\,(b_1,\dots,b_\ell) \sqsubseteq (a_1,\dots,a_\ell)\Big\}.$$

We can say that ρ_w is the least relation with the property

$$(b_1,\dots,b_\ell) \in \rho \ \wedge\ (b_1,\dots,b_\ell) \sqsubseteq (a_1,\dots,a_\ell) \ \Rightarrow\ (a_1,\dots,a_\ell) \in \rho_w.$$

We will also use the fact that if $(a_1,\dots,a_\ell) \in \rho_w$ and $(a_1,\dots,a_\ell) \sqsubseteq (c_1,\dots,c_\ell)$, then $(c_1,\dots,c_\ell) \in \rho_w$, because there is $(b_1,\dots,b_\ell) \in \rho$ such that $(b_1,\dots,b_\ell) \sqsubseteq (a_1,\dots,a_\ell)$ and thus, by transitivity of $\sqsubseteq$, $(b_1,\dots,b_\ell) \sqsubseteq (c_1,\dots,c_\ell)$.

Example 4.5.8 *If ρ is a binary relation on E_2 given by $\rho = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, then the weak extension of ρ is*

$$\rho_w = \begin{pmatrix} 0 & 0 & 0 & 2 & 2 & 2 \\ 0 & 1 & 2 & 0 & 1 & 2 \end{pmatrix}.$$

Definition 4.5.9 *An n-ary IS operation f w-*preserves *relation $\rho \in R_k^{(\ell)}$ (or ρ is w-*invariant *of f) if for every $\ell \times n$ matrix M in ρ^* it holds $f(M) \in \rho_w$.*

We denote by $wPol\,\rho$ the set of all IS operations that w-preserve relation ρ, and by $wInv\,f$ we denote the set of all relations that IS operation f w-preserves. Let us define the mappings

$$wPol : \mathcal{P}(R_k) \to \mathcal{P}(I_k) \quad \text{and} \quad wInv : \mathcal{P}(I_k) \to \mathcal{P}(R_k),$$

as follows:

$$wPol\,Q = \bigcap_{\rho \in Q} wPol\,\rho = \{f \in I_k : f \ w\text{-preserves each } \rho \in Q\},\ Q \subseteq R_k,$$
$$wInv\,F = \bigcap_{f \in F} wInv\,f = \{\rho \in R_k : \text{each } f \in F \ w\text{-preserves } \rho\},\ F \subseteq I_k.$$

Clearly, the pair $(wPol, wInv)$ is a Galois connection between relations and IS operations on E_k.

Lemma 4.5.10 *For every $\rho \subseteq E_k^\ell$ set $wPol\,\rho$ is an upward saturated IS clone.*

Proof. Since $Pol\,\rho \subseteq wPol\,\rho$, we have $J_k \subseteq wPol\,\rho$. Let $f \in (wPol\,\rho)^{(n)}$ and $g_1, \dots, g_n \in (wPol\,\rho)^{(m)}$. For an arbitrary $\ell \times m$ matrix $M \in \rho^*$ it holds

$$\big(g_j(M_{1*}), \dots, g_j(M_{\ell *})\big) \in \rho_w, \text{ for all } j = 1, \dots, n. \tag{4.1}$$

We also have

$$\big(f(g_1, \dots, g_n)\big)(M) = f^+ \begin{pmatrix} g_1(M_{1*}) & \cdots & g_n(M_{1*}) \\ \cdots & \cdots & \cdots \\ g_1(M_{\ell *}) & \cdots & g_n(M_{\ell *}) \end{pmatrix}$$
$$= \prod_{\substack{(y_{1j}, \dots, y_{\ell j}) \sqsubseteq (g_j(M_{1*}), \dots, g_j(M_{\ell *})) \\ j = 1, \dots, n}} f \begin{pmatrix} y_{11} & \cdots & y_{1n} \\ \cdots & \cdots & \cdots \\ y_{\ell 1} & \cdots & y_{\ell n} \end{pmatrix}$$

Condition 4.1 implies that there exists at least one matrix

$$M' = \begin{pmatrix} y_{11} & \cdots & y_{1n} \\ \cdots & \cdots & \cdots \\ y_{\ell 1} & \cdots & y_{\ell n} \end{pmatrix}$$

such that $(y_{1j}, \dots, y_{\ell j}) \sqsubseteq \big(g_j(M_{1*}), \dots, g_j(M_{\ell *})\big)$, for all $j = 1, \dots, n$, which is in ρ^* and thus $f(M') \in \rho_w$. Finally, using transitivity of the relation $\sqsubseteq$

and the fact that $a \sqsubseteq a \sqcap b$, we may conclude that $f(g_1, \ldots, g_n)(M) \in \rho_w$, i.e., $f(g_1, \ldots, g_n) \in wPol\,\rho$. Therefore, $wPol\,\rho$ is an IS clone.

To prove $wPol\,\rho$ is upward saturated we need to show that it contains all IS superoperations of its elements, i.e., if $f \in (wPol\,\rho)^{(n)}$ and $f \sqsubseteq g$, than $g \in wPol\,\rho$. For each $M \in \rho^*$ we have $f(M) = \big(f(M_{1*}), \ldots, f(M_{\ell*})\big) \in \rho_w$ and $\big(f(M_{1*}), \ldots, f(M_{\ell*})\big) \sqsubseteq \big(g(M_{1*}), \ldots, g(M_{\ell*})\big)$. Then by transitivity $\big(g(M_{1*}), \ldots, g(M_{\ell*})\big) \in \rho_w$, resulting in $g \in wPol\,\rho$. □

4.5.3 Extended IS operations and relations

Another way to investigate the correspondence between relations and IS operations is to observe relations on E_{k+1} and extended IS operations.

Let $\rho \subseteq E_{k+1}^{\ell}$. In general, $Pol\rho \cap I_k^+$ does not have to be an extended IS clone since it is not closed under Δ_i for $k \geq 2$ and it is not closed under $*_i$ for $k \geq 3$, as illustrated by the following example.

Example 4.5.11

(a) For $k = 2$ let $\rho = \{(1,0,2),(1,1,2)\}$ and $f^+, \Delta_i f^+$ be as follows:

f^+	0	1	2
0	1	0	2
1	0	1	2
2	2	2	2

	$\Delta_i f^+$
0	1
1	1
2	1

.

Then it is clear that $f^+ \in Pol\rho$ and $\Delta_i f^+ \notin Pol\rho$.

(b) For $k = 3$ let $\rho = \{(0,3),(3,0),(1,3),(3,1),(2,3),(3,2),(3,3)\}$, and consider the binary operation $f^+ \in Pol\rho$ and the unary operation $g^+ \in Pol\rho$, that are given by:

f^+	0	1	2	3
0	0	0	2	3
1	2	2	2	2
2	2	2	0	3
3	3	3	3	3

	g^+
0	2
1	2
2	1
3	3

Since $(2,3),(3,1) \in \rho$ and $\big((f^+ *_i g^+)(2,3),\ (f^+ *_i g^+)(3,\ 1)\big) = (2,2)$, it follows that $f^+ *_i g^+ \notin Pol\rho$.

We will introduce two classes of relations that induce closed sets of extended IS operations. With this aim, we prove the following properties.

Lemma 4.5.12 ([16]) *Let $f \in I_k^{(n)}$ and $g \in I_k^{(m)}$. Then*

(a) $\Delta_i f^+ \sqsubseteq \Delta f^+$ *and*

(b) $f^+ *_i g^+ \sqsubseteq f^+ * g^+$.

Proof.

(a) Let $a_1, \dots, a_{n-1} \in E_{k+1}$ and denote $(a_1, \dots, a_{n-1}), (b_1, \dots, b_{n-1})$ by $\vec{a}$ and $\vec{b}$, respectively. Then

$$\begin{aligned}
(\Delta f)^+(\vec{a}) &= \prod \left\{\Delta f(\vec{b}) : \vec{b} \in E_k^{n-1},\ \vec{b} \sqsubseteq \vec{a}\right\} \\
&= \prod \left\{f(b_1, b_1, \dots, b_{n-1}) : \vec{b} \in E_k^{n-1}, \vec{b} \sqsubseteq \vec{a}\right\} \\
&\sqsubseteq \prod \left\{f(b_1', b_1'', b_2, \dots, b_{n-1}) : (b_1', b_1'', b_2, \dots, b_{n-1}) \in E_k^n,\right. \\
&\qquad\qquad \left.(b_1', b_1'', b_2, \dots, b_{n-1}) \sqsubseteq (a_1, a_1, a_2, \dots, a_{n-1})\right\} \\
&= f^+(a_1, a_1, a_2, \dots, a_{n-1}) \\
&= \Delta f^+(\vec{a}).
\end{aligned}$$

(b) Let $a_1, \dots, a_{m+n-1} \in E_{k+1}$, and denote $(a_1, \dots, a_{m+n-1}), (b_1, \dots, b_{m+n-1})$ by $\vec{a}$ and $\vec{b}$, respectively. Then

$$\begin{aligned}
(f \diamond g)^+(\vec{a}) &= \prod \left\{(f \diamond g)(\vec{b}) : \vec{b} \in E_k^{m+n-1},\ \vec{b} \sqsubseteq \vec{a}\right\} \\
&= \prod \Big\{f^+\big(g(b_1, \dots, b_m), b_{m+1}, \dots, b_{m+n-1}\big) : \\
&\qquad\qquad \vec{b} \in E_k^{m+n-1},\ \vec{b} \sqsubseteq \vec{a}\Big\} \\
&\sqsubseteq f^+\big(g^+(a_1, \dots, a_m), a_{m+1}, \dots, a_{m+n-1}\big) \\
&= (f^+ * g^+)(\vec{a}).
\end{aligned}$$

□

Let $\rho \subseteq E_{k+1}^\ell$ satisfy the following property: $(a_1, \dots, a_\ell) \in \rho$ if and only if

$$\forall (b_1, \dots, b_\ell) \in E_k^\ell : (b_1, \dots, b_\ell) \sqsubseteq (a_1, \dots, a_\ell) \Rightarrow (b_1, \dots, b_\ell) \in \rho \tag{4.2}$$

Equivalently, $(a_1, \dots, a_\ell) \in \bar{\rho}$ if and only if

$$\exists (b_1, \dots, b_\ell) \in E_k^\ell : (b_1, \dots, b_\ell) \in \bar{\rho} \text{ and } (b_1, \dots, b_\ell) \sqsubseteq (a_1, \dots, a_\ell) \tag{4.3}$$

Evidently, ρ and $\bar{\rho}$ are strong and weak extension of some relations on E_k.

Lemma 4.5.13 ([16]) *Let $\rho \subseteq E_{k+1}^{\ell}$ satisfy property (4.2) and let $(x_1, \ldots, x_\ell)$, $(y_1, \ldots, y_\ell) \in E_{k+1}^{\ell}$ be such that $(x_1, \ldots, x_\ell) \sqsubseteq (y_1, \ldots, y_\ell)$.*

(a) If $(y_1, \ldots, y_\ell) \in \rho$, then $(x_1, \ldots, x_\ell) \in \rho$.

(b) If $(x_1, \ldots, x_\ell) \in \bar{\rho}$, then $(y_1, \ldots, y_\ell) \in \bar{\rho}$.

Proof. Straightforward from the properties (4.2) and (4.3) using transitivity of the relation $\sqsubseteq$. □

In the subsequent theorem we prove that whenever relation ρ on E_{k+1} satisfies property (4.2), then a set of extended IS operations preserving either relation ρ or its complement is a clone of extended IS operations. If we denote this set by C^+, then (by Corollary 3.2.4) C is a clone of incompletely specified operations.

Theorem 4.5.14 ([16]) *Let $\rho \subseteq E_{k+1}^{\ell}$ satisfy property (4.2). If*

(a) If $C^+ = Pol\rho \cap I_k^+$, then C is an IS clone.

(b) If $C^+ = Pol\bar{\rho} \cap I_k^+$, then C is an IS clone.

Proof. It is enough to prove that C^+ is closed with respect to Δ_i and $*_i$.

(a) Let $f^+ \in (Pol\rho)^{(n)}, g^+ \in (Pol\rho)^{(m)}$. Since $Pol\rho$ is a clone, it follows that $\Delta f^+, f^+ * g^+ \in Pol\rho$. If $A = (A_{1*}, \ldots, A_{\ell*})^T \in \rho^*_{n-1}$, then by Lemma 4.5.12 it holds

$$\big(\Delta_i f^+(A_{1*}), \ldots, \Delta_i f^+(A_{\ell*})\big) \sqsubseteq \big(\Delta f^+(A_{1*}), \ldots, \Delta f^+(A_{\ell*})\big) \in \rho.$$

Then using property (4.2) and Lemma 4.5.13 (a) we obtain

$$\big(\Delta_i f^+(A_{1*}), \ldots, \Delta_i f^+(A_{\ell*})\big) \in \rho.$$

Next, for any $A = (A_{1*}, \ldots, A_{\ell*})^T \in \rho^*_{m+n-1}$ we have

$$\begin{aligned}\big((f^+ *_i g^+)(A_{1*}), \ldots, (f^+ *_i g^+)(A_{\ell*})\big) &\sqsubseteq \\ \sqsubseteq \big((f^+ * g^+)(A_{1*}), \ldots, (f^+ * g^+)(A_{\ell*})\big) &\in \rho.\end{aligned}$$

Hence, by (4.2) and Lemma 4.5.13 (a),

$$\big((f^+ *_i g^+)(A_{1*}), \ldots, (f^+ *_i g^+)(A_{\ell*})\big) \in \rho.$$

(b) Let $f^+ \in (Pol\bar{\rho})^{(n)}$ and $A = (A_{1*}, \ldots, A_{\ell*})^T \in \bar{\rho}^*_{n-1}$. There is a matrix $B = (B_{1*}, \ldots, B_{\ell*})^T \in \bar{\rho}^*_{n-1}$ with all components from E_k such that $B_{1*} \sqsubseteq A_{1*}, \ldots, B_{\ell*} \sqsubseteq A_{\ell*}$. Since f^+ is monotone with respect to $\sqsubseteq$ we have

$$\big(\Delta_i f^+(A_{1*}), \ldots, \Delta_i f^+(A_{\ell*})\big) \sqsupseteq \big(\Delta_i f^+(B_{1*}), \ldots, \Delta_i f^+(B_{\ell*})\big).$$

For $B_{j*} \in E_k^{n-1}$, $j = 1, \ldots, \ell$, it holds

$$\Delta_i f^+(B_{j*}) = (\Delta f)^+(B_{j*}) = \Delta f(B_{j*}) = \Delta f^+(B_{j*}),$$

and using the fact that $f^+ \in Pol\bar{\rho}$ implies $\Delta f^+ \in Pol\bar{\rho}$, we get

$$\big(\Delta_i f^+(B_{1*}), \ldots, \Delta_i f^+(B_{\ell*})\big) = \big(\Delta f^+(B_{1*}), \ldots, \Delta f^+(B_{\ell*})\big) \in \bar{\rho}.$$

Now, by property (4.3) and Lemma 4.5.13 (b) it follows that

$$\big(\Delta_i f^+(A_{1*}), \ldots, \Delta_i f^+(A_{\ell*})\big) \in \bar{\rho}.$$

Let $f^+ \in (Pol\bar{\rho})^{(n)}$ and $g^+ \in (Pol\bar{\rho})^{(m)}$. Again for any matrix $A = (A_{1*}, \ldots, A_{\ell*}) \in \bar{\rho}^*_{m+n-1}$ there is a matrix $B = (B_{1*}, \ldots, B_{\ell*})^T \in \bar{\rho}^*_{m+n-1}$ with elements from E_k such that $B_{1*} \sqsubseteq A_{1*}, \ldots, B_{\ell*} \sqsubseteq A_{\ell*}$.

For $j = 1, \ldots, \ell$ we denote by B'_j the first m components of B_{j*} and by B''_j the last $n-1$ components. Hence, for $B_{j*} \in E_k^{m+n-1}$, $j = 1, \ldots, \ell$, it holds

$$(f^+ *_i g^+)(B_{j*}) = (f \diamond g)^+(B_{j*}) = (f \diamond g)(B_{j*}) = f^+\big(g(B'_j), B''_j\big),$$

and therefore, for $h^+ = f^+ *_i g^+$,

$$\begin{aligned}\big(h^+(A_{1*}), \ldots, h^+(A_{\ell*})\big) &\sqsupseteq \big(h^+(B_{1*}), \ldots, h^+(B_{\ell*})\big)\\ &= \Big(f^+\big(g(B'_1), B''_1\big), \ldots, f^+\big(g(B'_m), B''_m\big)\Big).\end{aligned}$$

Also $g(B'_j) = g^+(B'_j)$, and since $g^+ \in Pol\bar{\rho}$, we have

$$\big(g(B'_1), \ldots, g(B'_\ell)\big) = \big(g^+(B'_1), \ldots, g^+(B'_\ell)\big) \in \bar{\rho}.$$

Now, $f^+ \in Pol\bar{\rho}$ implies

$$\Big(f^+\big(g(B'_1), B''_1\big), \ldots, f^+\big(g(B'_\ell), B''_\ell\big)\Big) \in \bar{\rho}.$$

Finally, using property (4.3) and Lemma 4.5.13 (b) we obtain

$$\big((f^+ *_i g^+)(A_{1*}), \ldots, (f^+ *_i g^+)(A_{\ell*})\big) \in \bar{\rho}.$$

□

The following property is an immediate consequence of Lemma 4.5.13.

Corollary 4.5.15 *Let $\rho \subseteq E_{k+1}^{\ell}$ satisfy property (4.2).*

(a) If $f^+ \in Pol\rho$ and $g^+ \sqsubseteq f^+$ then $g^+ \in Pol\rho$.

(b) If $g^+ \in Pol\bar{\rho}$ and $g^+ \sqsubseteq f^+$ then $f^+ \in Pol\bar{\rho}$.

Chapter 5

Lattices on a two-element set

Lattice of clones on a two-element set was completely described by Emil Post in [49]. Its structure is not very complicated and it has countably many elements since it contains eight infinite chains. However, even on E_2 lattices of partial clones, hyperclones and IS clones (last two being isomorphic) are of continuum cardinality, and thus far more complex than Post lattice. Nevertheless, certain properties of these lattices are known and some of the results will be presented in the remainder of this chapter.

5.1 Lattice $\mathcal{L}_2$

In this section we will describe Post lattice in more detail. Notation, definitions of closed classes and the lattice diagram are taken from [5] (with a few slight changes). We use the usual symbols $\wedge$ (conjuction), $\vee$ (disjunction), $\oplus$ (addition modulo 2) and for negation we will use $\overline{x}$ instead of $\neg x$. Following shortened notation will also be used

$$h_n(x_1, \ldots, x_{n+1}) = \bigvee_{i=1}^{n+1} x_1 \wedge \ldots \wedge x_{i-1} \wedge x_{i+1} \wedge \ldots \wedge x_{n+1}$$
$$\mathsf{dual}(f)(a_1, \ldots, a_n) = \overline{f(\overline{a_1}, \ldots, \overline{a_n})}.$$

We now introduce some classes of Boolean functions that will be used in the definition of the clones on E_2. For $a \in E_2$, an n-ary Boolean function f is said to be

- *a-preserving* if $f(a, \ldots, a) = a$;

- *self-dual* if $f(a_1, \ldots, a_n) = \mathsf{dual}(f)(a_1, \ldots, a_n)$, for all $(a_1, \ldots, a_n) \in E_2^n$;
- *monotone* if $f(a_1, \ldots, a_n) \leq f(b_1, \ldots, b_n)$, for all $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in E_2^n$ such that $a_i \leq b_i$, $i = 1, \ldots, n$;
- *affine* if there exist $\alpha_0, \ldots, \alpha_n \in E_2$ such that $f(a_1, \ldots, a_n) = \alpha_0 \oplus (\alpha_1 \wedge a_1) \oplus \cdots \oplus (\alpha_n \wedge a_n)$, for all $(a_1, \ldots, a_n) \in E_2^n$;
- *a-separating* if there exists $i \in \{1, \ldots, n\}$ such that $f^{-1}(a) \subseteq E_2^{i-1} \times \{a\} \times E_2^{n-i}$;
- *a-separating of degree* m if for every $U \subseteq f^{-1}(a)$, with $|U| = m$, there exists $i \in \{1, \ldots, n\}$ such that $U \subseteq E_2^{i-1} \times \{a\} \times E_2^{n-i}$.

In the following table we present the list of all Boolean clones, each with its definition and one possible generating set.

Clone	Definition	Base
O_2	all Boolean functions	$\{x \wedge y, \overline{x}\}$
T_0	$\{f \mid f$ is 0-preserving$\}$	$\{x \wedge y, x \oplus y\}$
T_1	$\{f \mid f$ is 1-preserving$\}$	$\{x \vee y, x \oplus y \oplus 1\}$
T_2	$T_0 \cap T_1$	$\{x \vee y, x \wedge (y \oplus z \oplus 1)\}$
M	$\{f \mid f$ is monotone$\}$	$\{x \vee y, x \wedge y, c_0, c_1\}$
M_1	$M \cap T_1$	$\{x \vee y, x \wedge y, c_1\}$
M_0	$M \cap T_0$	$\{x \vee y, x \wedge y, c_0\}$
M_2	$M \cap T_2$	$\{x \vee y, x \wedge y\}$
S_0^n	$\{f \mid f$ is 0-separating of degree $n\}$	$\{x \Rightarrow y, \mathsf{dual}(h_n)\}$
S_0	$\{f \mid f$ is 0-separating$\}$	$\{x \Rightarrow y\}$
S_1^n	$\{f \mid f$ is 1-separating of degree $n\}$	$\{x \wedge \overline{y}, h_n\}$
S_1	$\{f \mid f$ is 1-separating$\}$	$\{x \wedge \overline{y}\}$
S_{02}^n	$S_0^n \cap T_2$	$\{x \vee (y \wedge \overline{z}), \mathsf{dual}(h_n)\}$
S_{02}	$S_0 \cap T_2$	$\{x \vee (y \wedge \overline{z})\}$
S_{01}^n	$S_0^n \cap M$	$\{\mathsf{dual}(h_n), c_1\}$
S_{01}	$S_0 \cap M$	$\{x \vee (y \wedge z), c_1\}$
S_{00}^n	$S_0^n \cap T_2 \cap M$	$\{x \vee (y \wedge z), \mathsf{dual}(h_n)\}$
S_{00}	$S_0 \cap T_2 \cap M$	$\{x \vee (y \wedge z)\}$

Clone	Definition	Base
S_{12}^n	$S_1^n \cap T_2$	$\{x \wedge (y \vee \overline{z}), h_n\}$
S_{12}	$S_1 \cap T_2$	$\{x \wedge (y \vee \overline{z})\}$
S_{11}^n	$S_1^n \cap M$	$\{h_n, c_0\}$
S_{11}	$S_1 \cap M$	$\{x \wedge (y \vee z), c_0\}$
S_{10}^n	$S_1^n \cap T_2 \cap M$	$\{x \wedge (y \vee z), h_n\}$
S_{10}	$S_1 \cap T_2 \cap M$	$\{x \wedge (y \vee z)\}$
S	$\{f \mid f$ is self-dual$\}$	$\{(x \wedge \overline{y}) \vee (x \wedge \overline{z}) \vee (\overline{y} \wedge \overline{z})\}$
D_1	$S \cap T_2$	$\{(x \wedge y) \vee (x \wedge \overline{z}) \vee (y \wedge \overline{z})\}$
D_2	$S \cap M$	$\{h_2\}$
L	$\{f \mid f$ is affine$\}$	$\{x \oplus y, c_1\}$
L_0	$L \cap T_0$	$\{x \oplus y\}$
L_1	$L \cap T_1$	$\{x \oplus y \oplus c_1\}$
L_2	$L \cap T_2$	$\{x \oplus y \oplus z\}$
L_3	$L \cap S$	$\{x \oplus y \oplus z \oplus c_1\}$
V	$\{f \mid f$ is a disjunction or a constant$\}$	$\{x \vee y, c_0, c_1\}$
V_0	$V \cap T_0$	$\{x \vee y, c_0\}$
V_1	$V \cap T_1$	$\{x \vee y, c_1\}$
V_2	$V \cap T_2$	$\{x \vee y\}$
Λ	$\{f \mid f$ is a conjunction or a constant$\}$	$\{x \wedge y, c_0, c_1\}$
Λ_0	$\Lambda \cap T_0$	$\{x \wedge y, c_0\}$
Λ_1	$\Lambda \cap T_1$	$\{x \wedge y, c_1\}$
Λ_2	$\Lambda \cap T_2$	$\{x \wedge y\}$
N	$\{f \mid f$ depends on at most one variable$\}$	$\{\overline{x}, c_0, c_1\}$
N_2	$N \cap T_2$	$\{\overline{x}\}$
I	$\{f \mid f$ is a projection or a constant$\}$	$\{c_0, c_1\}$
I_0	$I \cap T_0$	$\{c_0\}$
I_1	$I \cap T_1$	$\{c_1\}$
J_2	$I \cap T_2$	$\{e_1^1\}$

Table 5.1: List of all Boolean clones

The diagram of the Post lattice is given in Figure 5.1.

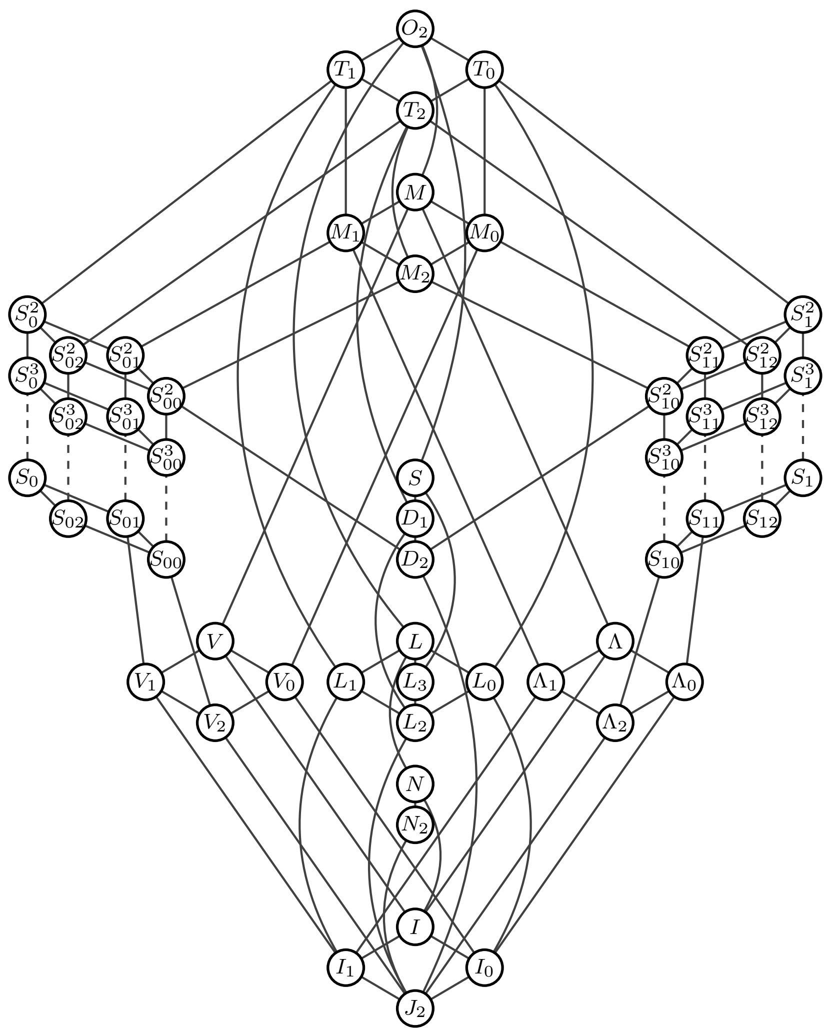

Figure 5.1: Post lattice

Clearly, there are 5 maximal clones (clones directly bellow O_2) in the lattice $\mathcal{L}_2$, and they are

$$T_0 = Pol(0), \qquad T_1 = Pol(1),$$

$$S = Pol\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad M = Pol\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \text{ and}$$

$$L = Pol\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

On the other hand, there are 7 minimal clones (clones directly above J_2). The minimal functions generating them are the following:

- constant functions c_0 and $c_1 \rightsquigarrow I_0$ and I_1
- negation $\overline{x} \rightsquigarrow N_2$
- conjunction $x \wedge y \rightsquigarrow \Lambda_2$
- disjunction $x \vee y \rightsquigarrow V_2$
- ternary majority function $(x \wedge y) \vee (x \wedge z) \vee (y \wedge z) \rightsquigarrow D_2$
- ternary linear (minority) function $x \oplus y \oplus z \rightsquigarrow L_2$

5.2 Lattice $\mathcal{L}_2^p$

Here we present the list of coatoms and atoms of the lattice of partial clones on E_2 and one result investigating position of the lattice $\mathcal{L}_2$ in $\mathcal{L}_2^p$.

Description of all maximal partial clones was given by Freivald, who also showed that there exist partial clones on E_2 which are not finitely generated.

Theorem 5.2.1 ([26]) *There are exactly 8 maximal partial clones on E_2, and they are the following:*

$$O_2 \cup \langle c_2 \rangle_p,\ pPOL(0),\ pPOL(1),\ pPOL\begin{pmatrix} 0 \\ 1 \end{pmatrix},\ pPOL\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix},$$

$$pPOL\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, pPOL\rho_1, pPOL\rho_2,$$

where

$$\rho_1 = \big\{(x,x,y,y) : x,y \in E_2\big\} \cup \big\{(x,y,y,x) : x,y \in E_2\big\} \text{ and}$$
$$\rho_2 = \rho_1 \cup \big\{(x,y,x,y) : x,y \in E_2\big\}.$$

There are 11 minimal partial clones [9] where 7 of them are total minimal clones and the minimal properly partial clones are generated by the following partial projections

	$e^1_{1,\emptyset}$	$e^1_{1,\{0\}}$	$e^1_{1,\{1\}}$
0	2	0	2
1	2	2	1

$e^2_{1,\{(0,0),(1,1)\}}$	0	1
0	0	2
1	2	1

For clone C on E_k let

$$\mathcal{I}(C) = \big\{D \subseteq P_k : D \text{ is partial clone such that } D \cap O_k = C\big\},$$

that is, $\mathcal{I}(C)$ is the set of all partial clones on E_k whose total part is clone C. It is known that the set $\mathcal{I}(C)$ is an interval in the lattice $\mathcal{L}_k^p$ (see [38]).

In [37] Lau posted the problem of describing the set $\mathcal{I}(C)$, where C is a total clone on E_2.

Some partial results were obtained in several papers and the classification was finalised in [39] and [20].

Theorem 5.2.2 ([20]) *Let C be a clone on E_2. Then the interval $\mathcal{I}(C)$ is finite if and only if*

$$C \in \big\{O_2, T_0, T_1, T_2, M, M_0, M_1, M_2, S, D_1\big\}.$$

Otherwise, it is of continuum cardinality.

Example 5.2.3 *Interval for the set of all total operations consists of 3 elements and it holds*

$$O_2 \subset O_2 \cup \langle c_2\rangle_p \subset P_2.$$

Example 5.2.4 *It is known that the clone* $T_0 = \{f \in O_2 : f(0, \dots, 0) = 0\}$ *is maximal. Let us define following sets of partial operations*

$$T_{0,0} = \{f \in P_2 : f(0, \dots, 0) = 0\},$$
$$T_{0,2} = \{f \in P_2 : f(0, \dots, 0) = 2\}.$$

Interval of T_0 *is given in Figure 5.2.*

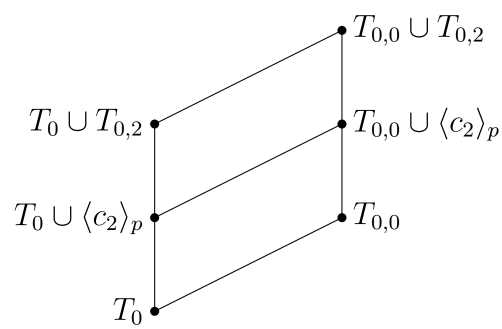

Figure 5.2: Hasse diagram of the interval $\mathcal{I}(T_0)$.

5.3 Lattice $\mathcal{L}_2^h$

As we previously mentioned, lattices $\mathcal{L}_2^h$ and $\mathcal{L}_2^{\text{IS}}$ are isomorphic by the following obvious isomorphism. Let the mapping $\eta : I_2 \to H_2$ be defined by $\eta(f) = f^*$, where

$$f^*(x_1, \dots, x_n) = \begin{cases} \{0\} & , f(x_1, \dots, x_n) = 0 \\ \{1\} & , f(x_1, \dots, x_n) = 1 \\ \{0,1\} & , f(x_1, \dots, x_n) = 2 \end{cases} .$$

Therefore, every property we have for hyperclones on E_2 also holds for IS clones on E_2.

H. Machida in [40] proved that the lattice $\mathcal{L}_2^h$ of all hyperclones on E_2 is of continuum cardinality. Since the cardinality of the set of hyperoperations on a two-element set is countable, the lattice of hyperclones has at most the cardinality of continuum. Therefore, this theorem is proved by constructing continuum many distinct hyperclones on E_2 in the following way.

We define the set $\mathcal{G} = \{g_n : n \geq 1\}$, where g_n is an n-ary hyperoperation on E_2 given by

$$g_n(x_1, \ldots, x_n) = \begin{cases} \{1\}, & \text{if } x_1 + \ldots + x_n \leq 1 \\ \{0,1\}, & \text{otherwise.} \end{cases}$$

It can be shown that for arbitrary $n \geq 1$ hyperoperation g_n cannot be generated by the remaining elements of $\mathcal{G}$. Consequently, distinct nonempty subsets of $\mathcal{G}$ generate different hyperclones, and since $\mathcal{G}$ is a countable set, it has continuum many subsets.

In [71], Tarasov introduced a different composition of partial functions on a two-element set. With this new definition, a partial function essentially coincide with a hyperoperation on E_2.

In [42], the authors adjust this description to the language of clones, by defining the set of all operations on $P^*_{E_2}$ that *r-preserves* $\rho \subseteq E_3^\ell$ as follows

$$rPol\,\rho = \Big\{ f \in O_{P^*_{E_2}} : \big(\delta(\{f\})\big)^{\#} \subseteq Pol\,\rho \Big\}.$$

If there is a relation $\rho_1 \subseteq E_2^\ell$ such that $\rho = \rho_1 \cup (E_3^\ell \setminus E_2^\ell)$, we will write $rPOL\,\rho_1$.

Theorem 5.3.1 ([71]) *There are nine maximal hyperclones on E_2. They are of the form $M_i' = M_i|_{\{0,1\}}$, $i = 1, \ldots, 9$, where*

$$M_1 = rPol(0\ \ 1),$$

$$M_2 = rPOL\big\{(x_1,x_2) \in E_2^2 : (x_1,x_2) \neq (1,0)\big\},$$

$$M_3 = rPOL\big\{(x_1,x_2) \in E_2^2 : (x_1,x_2) \notin \{(0,0),(1,1)\big\},$$

$$M_4 = rPOL\big\{(x_1,x_2,x_3,x_4) \in E_2^4 : x_1+x_2+x_3+x_4 = 0\big\},$$

$$M_5 = rPOL\big\{(x_1,x_2,x_3) \in E_2^3 : (x_1,x_2,x_3) \notin \{(0,1,1),(1,0,0)\}\big\},$$

$$M_6 = rPOL(0),$$

$$M_7 = rPOL(1),$$

$$M_8 = rPOL\big\{(x_1,x_2)\in E_2^2 : (x_1,x_2)\notin\{(0,0),(1,0),(1,1)\}\big\},$$
$$M_9 = rPOL\big\{(x_1,x_2,x_3,x_4)\in E_2^4 : x_1 = x_2 = x_3 = x_4 \\ \vee (x_2\neq x_3 \wedge x_1\neq x_4)\big\}.$$

The proof of the previous theorem consists of two parts:

(i) the proof that any set $C \subseteq H_2^{\#}$ such that $C \not\subseteq M_i$ for each $i \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is complete in $H_2^{\#}$, and

(ii) the proof that sets M_i, $1 \leq i \leq 9$, are pairwise disjoint. This was done by creating suitable representatives.

As for the atoms in the lattice $\mathcal{L}_2^h$, using the following facts:

(i) every minimal clone on E_k is also a minimal hyperclone on E_k,

(ii) for each hyperoperation f on E_2, set $\langle f\rangle_h$ is a minimal hyperclone on E_2 iff $\langle f^{\#}\rangle$ is a minimal clone on $P^*_{E_2}$ [45], and

(iii) description of all minimal clones on E_3 by Csákány (in [21]),

we can conclude that there are 13 minimal hyperclones on E_2.

Theorem 5.3.2 ([45]) *Each minimal hyperclone on E_2 is of the form $\langle f\rangle_h$, where f belongs to one of the following sets:*

$$Min^{(1)} = \{c_0, c_1, c_2, \overline{x}, f_0, f_1\};$$
$$Min^{(2)} = \{\mathtt{max}^h, \mathtt{min}^h, g_1, g_2, g_3\};$$
$$Min^{(3)} = \{\mathtt{ma}^h, \mathtt{mi}^h\},$$

where

$$c_0(x) = \{0\},\ c_1(x) = \{1\},\ c_2(x) = \{0,1\},$$
$$\overline{x} = \{\overline{x}\},\ f_0(x) = \{0,x\},\ f_1(x) = \{x,1\},$$
$$\mathtt{max}^h(x,y) = \{\mathtt{max}(x,y)\},\ \ \mathtt{min}^h(x,y) = \{\mathtt{min}(x,y)\},$$

x	y	g_1	g_2	g_3
0	0	$\{0\}$	$\{0\}$	$\{0\}$
0	1	$\{0,1\}$	$\{0\}$	$\{0,1\}$
1	0	$\{1\}$	$\{0,1\}$	$\{0,1\}$
1	1	$\{1\}$	$\{1\}$	$\{1\}$

$$\mathtt{ma}^h(x,y,z) = \{\mathtt{ma}(x,y,z)\} \ \ \textit{and} \ \ \mathtt{mi}^h(x,y,z) = \{\mathtt{mi}(x,y,z)\}.$$

Notice that hyperoperations $c_0, c_1, \overline{x}, \mathtt{max}^h, \mathtt{min}^h, \mathtt{ma}^h$ and $\mathtt{mi}^h$ correspond to 7 minimal total operations, while hyperoperations c_2, f_0, f_1 and g_3 correspond to 4 minimal properly partial operations. Only hyperoperations g_1 and g_2 generate essentially new atoms.

Chapter 6

Coatoms

Describing maximal elements of a lattice is of great importance since they are used in obtaining the completeness criterion.

We say that a set $F \subseteq O_k$ is *complete* if it generates the whole set O_k, that is $\langle F \rangle = O_k$. It is known that, since O_k is finitely generated, there are finitely many maximal clones on E_k and every proper subclone of O_k is contained in some maximal clone [74]. Therefore, set $F \subseteq O_k$ is complete if and only if it is not a subset of any of the maximal clones. The same holds for the lattices $\mathcal{L}_k^p, \mathcal{L}_k^{\mathrm{IS}}$ and $\mathcal{L}_k^h$.

All maximal clones of total and partial operations are described, but we are still far from getting the same result for hyperclones and IS clones. However, some classes of maximal clones of hyperoperations and incompletely specified operations are known.

6.1 Maximal clones

Definition 6.1.1 *Clone $M \subseteq O_k$ is* **maximal** *if for every clone C such that $M \subseteq C \subseteq O_k$ it holds $C = M$ or $C = O_k$.*

Clearly, clone M is maximal if and only if $M \neq O_k$ and for all $f \in O_k \setminus M$ it holds that $\langle M \cup \{f\} \rangle = O_k$.

One of the greatest achievements in clone theory so far is the complete classification of coatoms of the lattice $\mathcal{L}_k$, done by I.G. Rosenberg. And although it is a magnificent result, it was in fact the culmination of the joint effort of the number of mathematicians who in the 1950s and 1960s dealt with the problem of describing all maximal clones on a given set.

For one special case, Yablonskii showed in [74] that $Pol\,\iota_k^n$, where $\iota_k^n = \big\{(x_1,\dots,x_n) \in E_k^n : |\{x_1,\dots,x_n\}| \le n-1\big\}$, is a maximal clone, and moreover it is the only maximal clone that contains all unary operations on E_k. More generally, Kuznecov in [35] proved that every maximal clone is completely determined by a single relation, more precisely, each maximal clone is of the form $Pol\rho$ for some non-diagonal relation ρ. One might ask if this is the most precise characterization of the maximal clones we can achieve. Fortunately, the answer is no.

After Post, who by completely describing all clones on a two-element set also listed the 5 maximal ones, Yablonskii in [73] determined all 18 maximal clones on E_3, and reportedly Mal'tsev proved that there are exactly 82 maximal clones on E_4. Then in [55] Rosenberg described six classes of relations determining maximal clones on an arbitrary finite set (although most of the work was already done by Yablonskii in [74]) and eventually in [56] he proved that this list was complete.

Finally, we state Rosenberg's famous theorem.

Theorem 6.1.2 ([56]) *Let E_k be a finite set with $k \ge 2$. A clone on E_k is maximal if and only if it is of the form $Pol\ \rho$, where ρ is one of the following:*

(R_1) a bounded partial order, i.e., a partial order with the least and the greatest element;

(R_2) a prime permutation, i.e., the graph of a fixed point free permutation on E_k with all cycles of the same prime length;

(R_3) a prime affine relation, i.e., the graph of the ternary operation $x-y+z$ for some elementary Abelian p-group $(E_k;+,-,0)$ on E_k (p prime);

(R_4) a nontrivial equivalence relation, i.e., an equivalence relation that is neither the equality relation on E_k nor the full relation E_k^2;

(R_5) a central relation, i.e., $\emptyset \ne \rho \subsetneq E_k^\ell$, ρ is totally reflexive and totally symmetric and there exists an element $c \in E_k$ such that $\{c\} \times E_k^{\ell-1} \subseteq \rho$ (c is a central element of ρ, the set of all such elements is called center of ρ and is denoted by $C(\rho)$);

(R_6) *an ℓ-regular relation, $\ell \geq 3$. Let $\Theta = \{\theta_1, \ldots, \theta_h\}$, $h \geq 1$, be a family of equivalence relations such that each $\theta_i, 1 \leq i \leq h$, has exactly ℓ blocks, and for arbitrary blocks ε_i of θ_i, $1 \leq i \leq h$, it holds that $\bigcap\limits_{1\leq i\leq h} \varepsilon_i \neq \emptyset$. The relation*

$$\rho = \big\{(a_1, \ldots, a_\ell) \in E_k^\ell \,:\, \text{for every } 1 \leq i \leq h \text{ there are } 1 \leq j < k \leq \ell \text{ such that } (a_j, a_k) \in \theta_i\big\}$$

is said to be ℓ-regular.

The original proof is quite technical and difficult to follow. Probably the best known newer proof was done by Quackenbush [50], who proved that every maximal clone is of the form $Pol\rho$ for some of the Rosenberg's relations by using the connection between maximal clones and preprimal algebras. Similar proof was provided by Pinsker in [47].

6.2 Maximal partial clones

Set $O_k \cup \langle c_k \rangle_p$ consists of all total operations on E_k and all partial operations that are undefined for every input. We will now show that this set is a maximal partial clone. Moreover, it is the only maximal partial clone which contains O_k. The proof of the following lemma is more detailed version of the proof given in [68].

Lemma 6.2.1 *Set $O_k \cup \langle c_k \rangle_p$ is a maximal partial clone, i.e., for any $f \in P_k \setminus (O_k \cup \langle c_k \rangle_p)$ we have $\langle (O_k \cup \langle c_k \rangle_p) \cup \{f\} \rangle_p = P_k$.*

Proof. We already pointed out that $O_k \cup \langle c_k \rangle_p$ is a partial clone. Let us now prove that it is a coatom in the lattice of partial clones.

For an n-ary partial operation $f \in P_k \setminus (O_k \cup \langle c_k \rangle_p)$ there exist two n-tuples $(a_1, \ldots, a_n)$, $(b_1, \ldots, b_n) \in E_k^n$ such that

$$f(a_1, \ldots, a_n) \in E_k \quad \text{and} \quad f(b_1, \ldots, b_n) = k.$$

Let h be an arbitrary m-ary partial operation from P_k. For every $(x_1, \ldots, x_m) \in E_k^m$ we define operations $f_1, \ldots, f_n, g \in O_k^{(m)}$ in the following way

$$f_i(x_1, \ldots, x_m) = \begin{cases} a_i, & h(x_1, \ldots, x_m) \in E_k \\ b_i, & h(x_1, \ldots, x_m) = k \end{cases}$$

$$g(x_1, \ldots, x_m) = \begin{cases} h(x_1, \ldots, x_m), & h(x_1, \ldots, x_m) \in E_k \\ 0, & h(x_1, \ldots, x_m) = k \end{cases}$$

Now we can show that $h = e_1^2\big(g, f(f_1, \ldots, f_n)\big)$.

If $h(x_1, \ldots, x_m) \in E_k$, then $\big(f(f_1, \ldots, f_n)\big)(x_1, \ldots, x_m) = f(a_1, \ldots, a_n) \in E_k$. Hence,

$$\Big(e_1^2\big(g, f(f_1, \ldots, f_n)\big)\Big)(x_1, \ldots, x_m) = g(x_1, \ldots, x_m) = h(x_1, \ldots, x_m).$$

If $h(x_1, \ldots, x_m) = k$, then $\big(f(f_1, \ldots, f_n)\big)(x_1, \ldots, x_m) = f(b_1, \ldots, b_n) = k$ and $g(x_1, \ldots, x_m) = 0$, and thus,

$$\Big(e_1^2\big(g, f(f_1, \ldots, f_n)\big)\Big)(x_1, \ldots, x_m) = (e_1^2)_+(0, k) = k.$$

□

As we already mentioned, completeness problem for the Boolean partial functions was solved by Freivald in 1966 [26]. The same problem on E_3 was solved independently by Lau in [36] and Romov in [51], while the description of all coatoms of the lattice of partial clones on an arbitrary finite domain is due to Haddad and Rosenberg [28, 29, 30].

Before we state Haddad-Rosenberg theorem we need the following definitions. Notation and formulations used here are a combination of those used in [38] and [67]. In what follows we suppose that $1 \leq \ell \leq k$ unless stated otherwise.

Definition 6.2.2 *We define following quaternary relations on E_k:*

$$\rho_1 = \big\{(a, a, b, b), (a, b, a, b) \,:\, a, b \in E_k\big\},$$
$$\rho_2 = \big\{(a, a, b, b), (a, b, a, b), (a, b, b, a) \,:\, a, b \in E_k\big\}.$$

Definition 6.2.3 *Let Eq_ℓ be the set of all equivalence relations over $\{1,\dots,\ell\}$. For $\varepsilon \in Eq_\ell$, $\ell \geq 2$, we define*

$$\delta^{(\ell)}_{k,\varepsilon} = \big\{(a_1,\dots,a_\ell) \in E_k^\ell \,:\, (i,j) \in \varepsilon \Rightarrow a_i = a_j\big\}.$$

If k and ℓ are obvious from the context, we just write δ_ε.

Definition 6.2.4 *An ℓ-ary relation $\rho \subseteq E_k^\ell$ is called*

- areflexive *if $\rho \cap \delta_\varepsilon = \emptyset$ for every $\varepsilon \in Eq_\ell$, $\varepsilon \neq \iota^2_\ell$, i.e., for all $(x_1,\dots,x_\ell) \in \rho$ we have $x_i \neq x_j$ for all $1 \leq i < j \leq \ell$.*
- quasi-diagonal *if $\rho = \sigma \cup \delta_\varepsilon$, where σ is a nonempty areflexive relation, $\varepsilon \in Eq_\ell \setminus \{\iota^2_\ell\}$, and $\rho \neq E_k^2$ for $\ell = 2$.*

Definition 6.2.5 *Let $\rho \subseteq E_k^\ell$, $\sigma = \rho \setminus \iota_k^\ell$ and $\delta = \rho \cap \iota_k^\ell$. Let*

$$G_\sigma = \big\{\pi \in S_\ell \,:\, \sigma \cap \sigma^{[\pi]} \neq \emptyset\big\},$$

where S_ℓ is the set of all permutation over the set $\{1,\dots,\ell\}$ and $\sigma^{[\pi]} = \big\{(a_{\pi(1)},\dots,a_{\pi(\ell)}) \,:\, (a_1,\dots,a_\ell) \in \sigma\big\}$.
The model *of ρ is the ℓ-ary relation on $\{1,\dots,\ell\}$*

$$M(\rho) = \big\{(\pi(1),\dots,\pi(\ell)) \,:\, \pi \in G_\sigma\big\} \cup \big(\delta \cap \{1,\dots,\ell\}^\ell\big).$$

Definition 6.2.6 *Relation $\rho \subseteq E_k^\ell$ is* coherent *if the following conditions hold:*

1. *ρ is non-trivial relation;*
2. *(a) ρ is a unary relation, or*
   (b) ρ is areflexive, $2 \leq \ell \leq k$, or
   (c) ρ is quasi-diagonal, $2 \leq \ell \leq k$, or
   (d) $\delta = \iota_k^\ell$, $3 \leq \ell \leq k$, or
   (e) $\delta = \rho_i$, $i \in \{1,2\}$ (from Definition 6.2.2) and $\ell = 4$;
3. *$\sigma^{[\pi]} = \sigma$, for all $\pi \in G_\sigma$;*
4. *for every nonempty subset σ' of σ there exists a relational homomorphism $\varphi : E_k \to \{1,\dots,\ell\}$ from σ' to $M(\rho)$, such that $\big(\varphi(i_1),\dots,\varphi(i_\ell)\big) = (1,\dots,\ell)$, for some $(i_1,\dots,i_\ell) \in \sigma'$;*

5. (a) *if* $\delta = \iota_k^\ell$ *and* $\ell \geq 3$, *then* $G_\sigma = S_\ell$,
   (b) *if* $\delta = \rho_1$, *then* $G_\sigma = \langle (0231), (12) \rangle$,
   (c) *if* $\delta = \rho_2$, *then* $G_\sigma = S_4$.

Theorem 6.2.7 ([30]) *$M \subseteq P_k$ is a maximal partial clone iff $M = O_k \cup \langle c_k \rangle_p$ or $M = pPOL\rho$, where ρ is one of the following relations:*

- *an ℓ-ary ($1 \leq \ell \leq k$) non-trivial totally reflexive and totally symmetric relation;*

- *an ℓ-ary ($\ell \geq 2$) coherent areflexive or quasi-diagonal relation;*

- *a quaternary coherent relation $\sigma \cup \rho_i$, $i \in \{1, 2\}$, where σ is a nonempty quaternary areflexive relation.*

There are significantly more maximal partial clones than maximal clones on the same set. Let $\mathcal{M}_k$ and $p\mathcal{M}_k$ be the sets of maximal clones and maximal partial clones on E_k, respectively. In [67], using a computer program, Schölzel determined all elements of $p\mathcal{M}_5$ and $p\mathcal{M}_6$. The numbers of elements in $\mathcal{M}_k$ and $p\mathcal{M}_k$, for $2 \leq k \leq 6$, are presented in Table 6.1.

k	$\lvert\mathcal{M}_k\rvert$	$\lvert p\mathcal{M}_k\rvert$
2	5	8
3	18	58
4	82	1102
5	643	325722
6	15182	5242621816

Table 6.1: Sizes of $\mathcal{M}_k$ and $p\mathcal{M}_k$

6.3 Maximal hyperclones

In the lattice of hyperclones set of all total operations is a coatom, which was not the case for partial clones. This fact was firstly proved by Romov in [53] and here we give the idea of the proof from [22].

Lemma 6.3.1 ([53]) *Clone O_k of all total operations is a maximal hyperclone, i.e., for any $f \in H_k \setminus O_k$ we have $\langle O_k \cup \{f\}\rangle_h = H_k$.*

Sketch of a proof. For any n-ary hyperoperation $f \in H_k \setminus O_k$ there is at least one n-tuple $(a_1, \ldots, a_n) \in E_k^n$ such that $f(a_1, \ldots, a_n) = \{c_0, \ldots, c_{p-1}\}$, $p \geq 2$.

It is sufficient to prove that an arbitrary m-ary hyperoperation $h \in H_k$ can be constructed using (hyper)operations from $O_k \cup \{f\}$. We define maps $f_1, \ldots, f_n \in O_k^{(m)}$ and $g \in O_k^{(\ell+m)}$ in the following way.

If $h(y_1, \ldots, y_m) = \{d_0, d_1, \ldots, d_{q-1}\},\ q \geq 1$, then

$$\big(f_1(y_1, \ldots, y_m), \ldots, f_n(y_1, \ldots, y_m)\big) = \big(\{a_1\}, \ldots, \{a_n\}\big)$$

and

$$\begin{aligned} g(y_1, \ldots, y_m, c_0, \ldots, c_0, c_0) &= \{d_0\} \\ g(y_1, \ldots, y_m, c_0, \ldots, c_0, c_1) &= \{d_1\} \\ &\vdots \\ g(y_1, \ldots, y_m, c_{p-1}, \ldots, c_{p-1}, c_{p-1}) &= \{d_{q-1}\} \end{aligned}$$

where $\ell \in \mathbb{N}$ is such that $p^{\ell-1} < \max\limits_{(y_1,\ldots,y_m)\in A^m} |h(y_1, \ldots, y_m)| \leq p^{\ell}$.

Now, it can easily be shown that

$$h = g\big(e_1^m, \ldots, e_m^m, f(f_1, \ldots, f_n), \ldots, f(f_1, \ldots, f_n)\big),$$

which implies $h \in \langle O_k \cup f\rangle_h$.

Knowing that O_k is finitely generated, we conclude that the same holds for H_k. Consequently, there are finitely many maximal hyperclones and every hyperclone distinct from H_k is contained in some maximal hyperclone.

In [53] Romov also proved that for an arbitrary nonempty proper subset M of E_k, set of all hyperoperations with the property

$$(\forall a_1, \ldots, a_n \in M)\ f(a_1, \ldots, a_n) \cap M \neq \emptyset,$$

is a maximal hyperclone.

We will now present four classes of maximal hyperclones determined by some of the Rosenberg's relations.

The following theorem gives a sufficient condition for a hyperclone $hPol\rho$ to be maximal in the hyperclone lattice on E_k.

Theorem 6.3.2 ([41]) *Let $Pol\rho$ be a maximal clone on E_k such that*

$$(\forall f \in H_k \setminus hPol\rho)\,(\exists f' \in O_k \setminus Pol\rho)\ f' \in \langle Pol\rho \cup \{f\}\rangle_h. \tag{6.1}$$

Then $hPol\rho$ is a maximal hyperclone.

Sketch of a proof. Let ρ be a relation on E_k such that $Pol\rho$ is a maximal clone on E_k and let $f \in H_k \setminus hPol\rho$. Then we have an operation f' not in $Pol\rho$, generated by f and some operations from $Pol\rho$. Clearly, constant hyperoperation c_{E_k} is in $hPol\rho$ because $\rho \neq \emptyset$. Considering that $Pol\rho$ is a maximal clone and O_k is a maximal hyperclone, we obtain

$$\begin{aligned} H_k = \langle O_k \cup \{c_{E_k}\}\rangle_h &= \langle Pol\rho \cup \{f'\} \cup \{c_{E_k}\}\rangle_h \subseteq \\ &\subseteq \langle Pol\rho \cup \{f\} \cup \{c_{E_k}\}\rangle_h \subseteq \langle hPol\rho \cup \{f\}\rangle_h \subseteq H_k, \end{aligned}$$

implying $\langle hPol\rho \cup \{f\}\rangle_h = H_k$, i.e., $hPol\rho$ is a maximal hyperclone.

Subsequent theorems are all proved by constructing the operation f' from Theorem 6.3.2.

6.3.1 Hyperclones determined by bounded partial orders

In this section we will show that for every bounded partial order ρ hyperclone $hPol\rho$ is maximal because it satisfies the conditions of Theorem 6.3.2, that is for every hyperoperation f which is not in $hPol\rho$ we are going to construct an operation f', which does not preserve ρ, as a composition of some operations from $Pol\rho$ and the hyperoperation f.

Let $\rho \subseteq E_k^2$ be a bounded partial order, with the least element $\mathbf{0}$, and the greatest element $\mathbf{1}$. Next we choose $B, C \in P^*_{E_k}$ such that $(B \times C) \cap \rho = \emptyset$. Since ρ is reflexive, it holds $B \cap C = \emptyset$.

Let us define the following sets:

$$B' = \Big\{x \in E_k : (b', x) \in \rho \text{ and } (x, b'') \in \rho \ \text{ for some } \ b', b'' \in B\Big\},$$

$$C' = \Big\{x \in E_k : (c', x) \in \rho \text{ and } (x, c'') \in \rho \ \text{ for some } \ c', c'' \in C\Big\},$$

$$C'' = \Big\{x \in E_k : x \notin B' \text{ and } (c', x) \in \rho \text{ and } (x, b') \in \rho \ \text{ for some } \ b' \in B \text{ and } c' \in C\Big\},$$

$$G = \Big\{x \in E_k : x \notin B' \cup C' \cup C'' \text{ and } \big((b', x) \in \rho \ \text{ for some } \ b' \in B \text{ or } \ (c', x) \in \rho \text{ for some } c' \in C' \cup C''\big)\Big\},$$

$$L = \Big\{x \in E_k : x \notin B' \cup C' \cup C'' \text{ and } \big((x, b') \in \rho \ \text{ for some } \ b' \in B \text{ or } \ (x, c') \in \rho \ \text{ for some } \ c' \in C' \cup C''\big)\Big\}.$$

Lemma 6.3.3 ([17]) *For all $b \in B'$ and $c \in C' \cup C''$ it holds $(b, c) \notin \rho$.*

Proof. Suppose that there exist $b \in B'$ and $c \in C' \cup C''$ such that $(b, c) \in \rho$. There are two possibilities.

- If $c \in C'$, then for some $b' \in B$ and $c' \in C$ we have $(b', b), (b, c), (c, c') \in \rho$. Hence, by the transitivity of relation ρ, $(b', c') \in \rho$, which contradicts the choice of the sets B and C.
- If, on the other hand, $c \in C''$, then there exist $b', b'' \in B$ such that $(b', b), (b, c), (c, b'') \in \rho$, thus $(b', c), (c, b'') \in \rho$. However, this would imply $c \in B'$, which is impossible since $B' \cap C'' = \emptyset$.

□

Choose arbitrary $b \in B$ and $c \in C$ if $(C \times B) \cap \rho = \emptyset$, or choose $(b, c) \in B \times C$ such that $(c, b) \in \rho$, otherwise. Let us define the unary operation $g^{\rho}_{b,c}$ as follows

$$g^{\rho}_{b,c}(x) = \begin{cases} b, & x \in B' \\ c, & x \in C' \cup C'' \\ \mathbf{1}, & x \in G \\ \mathbf{0}, & x \in L \\ x, & \text{otherwise.} \end{cases} \tag{6.2}$$

Lemma 6.3.4 ([17]) *Operation $g^{\rho}_{b,c} \in O^{(1)}_k$ is well defined.*

Proof. We should prove that the sets $B', C' \cup C'', G$ and L are pairwise disjoint. As G and L by definition satisfy

$$(G \cup L) \cap (B' \cup C' \cup C'') = \emptyset,$$

we only have to show that $B' \cap (C' \cup C'') = \emptyset$ and $G \cap L = \emptyset$.
B' and C'' are disjoint by definition of C''. Suppose that there is $x \in B' \cap C'$. Then we have $(c', x), (x, b') \in \rho$, for some $b' \in B$ and $c' \in C$, and therefore $x \in C''$, which is not possible.
If $x \in G \cap L$, there are $b', b'' \in B$ and $c', c'' \in C' \cup C''$ such that

$$\big((b', x) \in \rho \vee (c', x) \in \rho\big) \wedge \big((x, b'') \in \rho \vee (x, c'') \in \rho\big).$$

We discuss the following cases:

i) $(b', x), (x, b'') \in \rho$, implying $x \in B'$, which is impossible since $(G \cup L)$ and B' are disjoint sets;

ii) $(b', x), (x, c'') \in \rho$, implying $(b', c'') \in \rho$, for $b' \in B \subseteq B'$ and $c'' \in C' \cup C''$, which is not possible by Lemma 6.3.3;

iii) $(c', x), (x, b'') \in \rho$. For $c' \in C' \cup C''$ there is $c_1 \in C$ such that $(c_1, c') \in \rho$, i.e., $(c_1, x), (x, b'') \in \rho$, for some $b'' \in B$ and $c_1 \in C$. Therefore, $x \in C''$, giving a contradictions with the definition of G and L;

iv) $(c', x), (x, c'') \in \rho$. For $c' \in C' \cup C''$ there is $c_1 \in C$ such that $(c_1, c') \in \rho$. If $c'' \in C'$, we will take $c_2 \in C$ such that $(c'', c_2) \in \rho$, and if $c'' \in C''$, we will take $b_2 \in B$ such that $(c'', b_2) \in \rho$. In both cases $x \in C' \cup C''$, which gives a contradiction with the definition of G and L.

□

Operation $g^{\rho}_{b,c}$ is one of the operations that we are going to use for the construction of the operation $f' \in O_k \setminus Pol\rho$, and therefore it is essential that $g^{\rho}_{b,c}$ preserves ρ, which we will prove next.

Lemma 6.3.5 ([17]) *Let $g^{\rho}_{b,c}$ be defined by (6.2). Then $g^{\rho}_{b,c} \in Pol\rho$.*

Proof. Let $x, y \in E_k$. We distinguish the following 25 cases:

$x\backslash y$	B'	$C' \cup C''$	G	L	other
B'	1	2	3	4	5
$C' \cup C''$	6	7	8	9	10
G	11	12	13	14	15
L	16	17	18	19	20
other	21	22	23	24	25

We are going to show that in each of the cases $(x, y) \notin \rho$, or $(x, y) \in \rho$ implies $\big(g^{\rho}_{b,c}(x), g^{\rho}_{b,c}(y)\big) \in \rho$.

In the following table we denote by ✓ all the cases with $(x, y) \in \rho$ implying $\big(g^{\rho}_{b,c}(x), g^{\rho}_{b,c}(y)\big) \in \rho$, and by $\times$ all those cases in which assumption that $(x, y) \in \rho$ leads to contradiction.

$x\backslash y$	B'	$C' \cup C''$	G	L	other
B'	✓	$\times$	✓	$\times$	$\times$
$C' \cup C''$	✓	✓	✓	$\times$	$\times$
G	$\times$	$\times$	✓	$\times$	$\times$
L	✓	✓	✓	✓	✓
other	$\times$	$\times$	✓	$\times$	✓

1) Let $x, y \in B'$ and suppose that $(x, y) \in \rho$. Then $\big(g^{\rho}_{b,c}(x), g^{\rho}_{b,c}(y)\big) = (b, b) \in \rho$, since ρ is reflexive.

2) If $x \in B'$ and $y \in C' \cup C''$, then $(x, y) \notin \rho$ by Lemma 6.3.3.

3) For $x \in B'$ and $y \in G$, if $(x, y) \in \rho$, then $\big(g^{\rho}_{b,c}(x), g^{\rho}_{b,c}(y)\big) = (b, \mathbf{1}) \in \rho$, since $\mathbf{1}$ is the greatest element. Let $x \in B'$ and $y \in L$, and assume that $(x, y) \in \rho$. We have the following possibilities:

   (a) There is $c' \in C' \cup C''$ such that $(y, c') \in \rho$. Then $(x, c') \in \rho$ for $x \in B'$ and $c' \in C' \cup C''$, which is not possible by Lemma 6.3.3.

   (b) There are $b', b'' \in B$ such that $(b', x), (x, y), (y, b'') \in \rho$, which means that $y \in B'$, contradicting the fact that $B' \cap L = \emptyset$.

5) In case $x \in B'$ and $y \notin B' \cup C' \cup C'' \cup G \cup L$ we would have that $(x, y) \in \rho$ implies $y \in G$.

6) If $x \in C' \cup C''$ and $y \in B'$, then $(c', x), (x, y), (y, b') \in \rho$, and therefore $(c', b') \in \rho$, for some $c' \in C$ and $b' \in B$. Hence, $(C \times B) \cap \rho \neq \emptyset$ implies $\big(g^{\rho}_{b,c}(x), g^{\rho}_{b,c}(y)\big) = (c, b) \in \rho$.

7) From $(x, y) \in \rho$ for $x, y \in C' \cup C''$ it holds that $\big(g^{\rho}_{b,c}(x), g^{\rho}_{b,c}(y)\big) = (c, c) \in \rho$, since ρ is reflexive.

8) For $x \in C' \cup C''$ and $y \in G$, as in the case (3), $(x, y) \in \rho$ implies $\big(g^{\rho}_{b,c}(x), g^{\rho}_{b,c}(y)\big) = (c, \mathbf{1}) \in \rho$.

9) If $x \in C' \cup C'', y \in L$ and $(x, y) \in \rho$, we consider the following cases:

   (a) $(c', x), (x, y), (y, b') \in \rho$, for some $b' \in B$ and $c' \in C$, meaning that $(c', y), (y, b') \in \rho$, for $b' \in B$ and $c' \in C$, i.e., $y \in C''$.

   (b) $(c', x), (x, y), (y, c'') \in \rho$, for some $c' \in C$ and $c'' \in C' \cup C''$. Then $(c', y), (y, c'') \in \rho$ for $c' \in C$ and $c'' \in C' \cup C''$ and therefore $y \in C' \cup C''$.

10) Let $x \in C' \cup C''$ and $y \notin B' \cup C' \cup C'' \cup G \cup L$. Then $(x, y) \in \rho$ would imply $y \in G$.

11) Let us suppose that $(x, y) \in \rho$ for $x \in G$ and $y \in B'$. We have:

   (a) There are $b', b'' \in B$ such that $(b', x), (x, y), (y, b'') \in \rho$, and so $x \in B'$.

   (b) There are $c' \in C' \cup C''$ and $b'' \in B$ such that $(c', x), (x, y), (y, b'') \in \rho$, i.e., $(c'', c'), (c', x), (x, b'') \in \rho$, for some $c'' \in C, c' \in C' \cup C''$ and $b'' \in B$. Thus $x \in C''$.

12) As $x \in G \wedge y \in C' \cup C'' \Leftrightarrow (x \in G \wedge y \in C') \vee (x \in G \wedge y \in C'')$, we distinguish the following four cases:

   (a) $(b', x), (x, y), (y, c'') \in \rho$, for some $b' \in B$ and $c'' \in C$, i.e., $(b', c'') \in \rho$, for $b' \in B$ and $c'' \in C$, which is not possible by the choice of B and C.

   (b) There are $c' \in C' \cup C''$ and $c'' \in C$ such that $(c', x), (x, y), (y, c'') \in \rho$. Hence, there is $c''' \in C$ such that $(c''', c') \in \rho$, i.e., $(c''', x), (x, c'') \in \rho$, for some $c'', c''' \in C$, which implies $x \in C'$.

   (c) It holds $(b', x), (x, y), (y, b'') \in \rho$, for some $b', b'' \in B$, and therefore $x \in B'$.

(d) There are $c' \in C' \cup C''$ and $b'' \in B$ such that $(c', x), (x, y), (y, b'') \in \rho$. Then $x \in C''$.

13) If $x, y \in G$ and $(x, y) \in \rho$, then $\big(g^{\rho}_{b,c}(x), g^{\rho}_{b,c}(y)\big) = (\mathbf{1}, \mathbf{1}) \in \rho$.

14) Let $x \in G$ and $y \in L$. Then, by the definition of G and L, the following cases are possible:

(a) There are $b', b'' \in B$, such that $(b', x), (x, y), (y, b'') \in \rho$, and therefore $x \in B'$.

(b) It holds $(b', x), (x, y), (y, c'') \in \rho$ for some $b' \in B$ and $c'' \in C' \cup C''$, implying that $(b', c'') \in \rho$, for $b' \in B'$ and $c'' \in C' \cup C''$ which is not possible by Lemma 6.3.3.

(c) $(c', x), (x, y), (y, b'') \in \rho$, for some $c' \in C' \cup C''$ and $b'' \in B$, implies $x \in C''$.

(d) There are $c', c'' \in C' \cup C''$ such that $(c', x), (x, y), (y, c'') \in \rho$. From this condition, we have $x \in C' \cup C''$.

15) For $x \in G$ and $y \notin B' \cup C' \cup C'' \cup G \cup L$, from $(x, y) \in \rho$ it follows that $y \in G$.

16) If $x \in L$ and $y \in B'$, then $(x, y) \in \rho$ and $\big(g^{\rho}_{b,c}(x), g^{\rho}_{b,c}(y)\big) = (\mathbf{0}, b) \in \rho$, since $\mathbf{0}$ is the least element.

17) Similarly as in the previous case, $x \in L$ and $y \in C' \cup C''$ imply $(x, y) \in \rho \Rightarrow \big(g^{\rho}_{b,c}(x), g^{\rho}_{b,c}(y)\big) = (\mathbf{0}, c) \in \rho$.

18) If $x \in L, y \in G$ and $(x, y) \in \rho$, then $\big(g^{\rho}_{b,c}(x), g^{\rho}_{b,c}(y)\big) = (\mathbf{0}, \mathbf{1})$, which is obviously in ρ.

19) In case $(x, y) \in \rho$ for $x, y \in L$, we have $\big(g^{\rho}_{b,c}(x), g^{\rho}_{b,c}(y)\big) = (\mathbf{0}, \mathbf{0}) \in \rho$.

20) Let $x \in L$ and $y \notin B' \cup C' \cup C'' \cup G \cup L$. Then, as in cases (16) and (17), from $(x, y) \in \rho$ we obtain $\big(g^{\rho}_{b,c}(x), g^{\rho}_{b,c}(y)\big) = (\mathbf{0}, y) \in \rho$.

21) For $x \notin B' \cup C' \cup C'' \cup G \cup L$ and $y \in B'$ the assumption $(x, y) \in \rho$ implies $x \in L$.

22) As in the previous case, from $(x, y) \in \rho$ for $x \notin B' \cup C' \cup C'' \cup G \cup L$ and $y \in C' \cup C''$, we have $x \in L$.

23) If $x \notin B' \cup C' \cup C'' \cup G \cup L$ and $y \in G$, as in the cases (3) and (8), from $(x, y) \in \rho$ we get $\big(g^{\rho}_{b,c}(x), g^{\rho}_{b,c}(y)\big) = (x, \mathbf{1}) \in \rho$.

24) For $x \notin B' \cup C' \cup C'' \cup G \cup L$ and $y \in L$, $(x, y) \in \rho$ implies $x \in L$.

25) If $(x, y) \in \rho$ for $x, y \notin B' \cup C' \cup C'' \cup G \cup L$, it holds $\big(g^{\rho}_{b,c}(x),\, g^{\rho}_{b,c}(y)\big) = (x, y) \in \rho$.

□

Now we can prove that if ρ is a bounded partial order, then $hPol\rho$ satisfies the conditions of the Theorem 6.3.2.

Theorem 6.3.6 ([17]) *Let $\rho \subset E_k^2$ be a non-trivial bounded partial order relation on E_k. Then $hPol\rho$ is a maximal hyperclone on E_k.*

Proof. Let $f \in H_k \setminus hPol\rho$ be an n-ary hyperoperation. Then there exists a matrix

$$M = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix} \in \rho^*$$

such that $f(M) = (B, C)$, where $(b, c) \notin \rho$, for all $b \in B$ and $c \in C$.

For every $i \in \{1, \dots, n\}$ define $f_i \in O_k^{(1)}$ as follows

$$f_i(x) = \begin{cases} a_i, & x = \mathbf{0} \\ b_i, & \text{otherwise.} \end{cases}$$

We shall prove that $f_1, \dots, f_n \in Pol\rho$. For $(x, y) \in \rho$ we get

$$\big(f_i(x), f_i(y)\big) = \begin{cases} (a_i, a_i), & \text{for } x = y = \mathbf{0} \\ (a_i, b_i), & \text{for } x = \mathbf{0} \text{ and } y \neq \mathbf{0} \\ (b_i, b_i), & \text{for } x \neq \mathbf{0} \text{ and } y \neq \mathbf{0}, \end{cases}$$

hence $\big(f_i(x), f_i(y)\big) \in \rho$ for all $i \in \{1, \dots, n\}$.

Let $g^{\rho}_{b,c}$ be the unary operation defined by (6.2). We proved (Lemma 6.3.5) that $g^{\rho}_{b,c} \in Pol\rho$, and therefore we can define operation $f' \in \langle Pol\rho \cup \{f\}\rangle_h$ by

$$f' = g^{\rho}_{b,c}\big(f(f_1, \dots, f_n)\big).$$

Although $f_1, \ldots, f_n$ and $g^{\rho}_{b,c}$ are total operations, when we compose them with a hyperoperation, we consider them to be hyperoperations with the singleton output values. Then for all $x \in E_k$ we have

$$f'(x) = g^{\rho}_{b,c}\big(f(f_1, \ldots, f_n)\big)(x) = (g^{\rho}_{b,c})^{\#}\Big(f^{\#}\big(f_1(x), \ldots, f_n(x)\big)\Big).$$

From this definition and since $B \subseteq B'$ and $C \subseteq C'$ trivially holds, we obtain

$$\begin{aligned} f'(\mathbf{0}) &= (g^{\rho}_{b,c})^{\#}\Big(f^{\#}\big(f_1(\mathbf{0}), \ldots, f_n(\mathbf{0})\big)\Big) = (g^{\rho}_{b,c})^{\#}\Big(f^{\#}\big(\{a_1\}, \ldots, \{a_n\}\big)\Big) \\ &= (g^{\rho}_{b,c})^{\#}\big(f(a_1, \ldots, a_n)\big) = (g^{\rho}_{b,c})^{\#}(B) = \{b\}, \\ f'(x) &= (g^{\rho}_{b,c})^{\#}\Big(f^{\#}\big(f_1(x), \ldots, f_n(x)\big)\Big) = (g^{\rho}_{b,c})^{\#}\Big(f^{\#}\big(\{b_1\}, \ldots, \{b_n\}\big)\Big) \\ &= (g^{\rho}_{b,c})^{\#}\big(f(b_1, \ldots, b_n)\big) = (g^{\rho}_{b,c})^{\#}(C) = \{c\}, \quad x \neq \mathbf{0}. \end{aligned}$$

Since $\mathbf{0}$ is the least element, it follows that $(\mathbf{0}, x) \in \rho$ for every $x \in A$. For $x \neq \mathbf{0}$ we have $\big(f'(\mathbf{0}), f'(x)\big) = (b, c) \notin \rho$. Hence, $f' \in O_k$ and $f' \notin Pol\rho$.

Therefore, conditions of the Theorem 6.3.2 are satisfied and we may conclude that $hPol\rho$ is a maximal hyperclone for every bounded partial order ρ. □

Example 6.3.7 *Let $\rho \subseteq E_5^2$ be a non-linear order given by*

$$\rho = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 4 & 4 & 4 \end{pmatrix}.$$

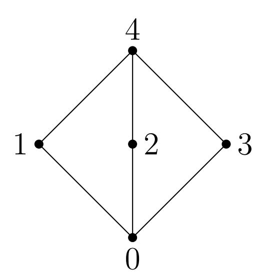

Figure 6.1: Hasse diagram of the partialy ordered set (E_5, ρ).

If we take $M = \begin{pmatrix} 0 & 3 \\ 2 & 4 \end{pmatrix} \in \rho^*$ *and* $f \in H_5^{(2)}$ *such that* $f(M) = (\{1\}, \{2,3\}) = (B, C) \notin \rho_h$, *we can define operations* $f_1, f_2 \in O_5^{(1)}$ *as follows*

	0	1	2	3	4
f_1	0	2	2	2	2
f_2	3	4	4	4	4

Sets used for defining the auxiliary operation $g_{b,c}^{\rho}$ *in this case are*

$$B' = \{1\},\ C' \cup C'' = \{2,3\},\ L = \{0\},\ G = \{4\},$$

and if we choose $b = 1$, $c = 2$, *we get*

	0	1	2	3	4
$g_{1,2}^{\rho}$	0	1	2	2	4

It is easily verified that $f_1, f_2, g_{1,2}^{\rho} \in Pol\rho$ *and for the operation* $f' = g_{1,2}^{\rho}\big(f(f_1, f_2)\big)$ *we have*

$$\begin{aligned} f'(0) &= g_{1,2}^{\rho}\big(f(f_1,f_2)\big)(0) = (g_{1,2}^{\rho})^{\#}\Big(f^{\#}\big(f_1(0), f_2(0)\big)\Big) \\ &= (g_{1,2}^{\rho})^{\#}\big(f^{\#}(\{0\},\{3\})\big) = (g_{1,2}^{\rho})^{\#}(\{1\}) = \{1\}, \\ f'(2) &= g_{1,2}^{\rho}\big(f(f_1,f_2)\big)(2) = (g_{1,2}^{\rho})^{\#}\Big(f^{\#}\big(f_1(2), f_2(2)\big)\Big) \\ &= (g_{1,2}^{\rho})^{\#}\big(f^{\#}(\{2\},\{4\})\big) = (g_{1,2}^{\rho})^{\#}(\{2,3\}) = \{2\}. \end{aligned}$$

Since $(0,2) \in \rho$ *and* $\big(f'(0), f'(1)\big) = (1,2) \notin \rho$, *we conclude that* $f' \notin Pol\rho$. *Finally, applying Theorem 6.3.2 we get* $\langle hPol\rho \cup \{f\}\rangle_h = H_5$.

6.3.2 Hyperclones determined by equivalence relations

In this section we again prove that $hPol\,\rho$ is a maximal hyperclone if ρ is a nontrivial equivalence relation by showing that the conditions of Theorem 6.3.2 are fulfilled.

Theorem 6.3.8 ([41]) *Let* $\rho \subseteq E_k^2$ *be a nontrivial equivalence relation. Then* $hPol\rho$ *is a maximal hyperclone on* E_k.

Sketch of a proof. Suppose that $f \in H_k \setminus hPol\rho$. Then there exists a matrix

$$M = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ b_1 & b_2 & \dots & b_n \end{pmatrix} \in \rho^*$$

such that $f(M) = (A', B') \notin \rho_h$, i.e., $(A' \times B') \cap \rho = \emptyset$.

For $i = 1, \dots, n$ we define operations $f_i \in O_k^{(1)}$ by

$$f_i(x) = \begin{cases} a_i, & x = a \\ b_i, & x = b \\ a_i, & \text{otherwise} \end{cases}$$

where $a = a_j$ and $b = b_j$, for some $j \in \{1, \dots, n\}$, such that $a_j \neq b_j$.

We also define operation $g \in O_k^{(1)}$ as follows

$$g(x) = \begin{cases} a', & x \in C_q, \text{ for some } \ q \in A' \\ b', & x \in C_q, \text{ for some } \ q \in B' \\ a', & \text{otherwise,} \end{cases}$$

where C_q is an equivalence class of the relation ρ containing element q, and a' and b' are arbitrary elements of A' and B' respectively.

Since $f_1, \dots, f_n, g \in Pol\rho$, it is now possible to define $f' \in \langle Pol\rho \cup \{f\} \rangle_h$ by

$$f' = g\big(f(f_1, \dots, f_n)\big).$$

For $(a, b) \in \rho$ we have $\big(f'(a), f'(b)\big) = (a', b') \notin \rho$, implying $f' \notin Pol\rho$.

Finally, using Theorem 6.3.2 we may conclude that $hPol\rho$ is a maximal hyperclone.

Example 6.3.9 *For an equivalence relation ρ on E_3 with classes $\{0\}$ and $\{1, 2\}$, i.e.,*

$$\rho = \begin{pmatrix} 0 & 1 & 2 & 1 & 2 \\ 0 & 1 & 2 & 2 & 1 \end{pmatrix},$$

we can choose $M = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \in \rho^*$ and a binary hyperoperation f such that $f(M) = (\{0\}, \{1, 2\}) = (A', B') \notin \rho_h$, so that $f \notin hPol\rho$. If we select $a = 1$, $b = 2$ and also $a' = 0$, $b' = 1$, we can define $f_1, f_2, g \in O_3^{(1)}$ by

	0	1	2
f_1	0	0	0
f_2	1	1	2
g	0	1	1

For the operation $f' = g\big(f(f_1, f_2)\big)$ *it holds* $(1,2) \in \rho$ *and* $\big(f'(1), f'(2)\big) = (0,1) \notin \rho$, *hence* $f' \notin Pol\rho$.

6.3.3 Hyperclones determined by central relations

Using the same general idea as in the previous cases we will prove that $hPol\,\rho$ is a maximal hyperclone whenever ρ is a central relation, i.e., totally reflexive and totally symmetric relation with a non-empty center.

Theorem 6.3.10 ([41]) *Let* $\rho \subseteq E_k^\ell$, $\ell \geq 1$, *be a central relation. Then* $hPol\rho$ *is a maximal hyperclone on* E_k.

Sketch of a proof. If we take $f \notin hPol\rho$, then we have a matrix

$$M = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{\ell 1} & a_{\ell 2} & \dots & a_{\ell n} \end{pmatrix} \in \rho^*$$

such that $f(M) = (A_1, A_2, \dots, A_\ell) \notin \rho_h$. Total reflexivity of the relation ρ implies that the sets $A_1, \dots, A_\ell$ are pairwise disjoint.

We choose distinct elements $c \in C(\rho)$ and $b_2, \dots, b_\ell \in E_k$ and define operations $f_i \in O_k^{(1)}$, $i = 1, \dots, n$, as follows

$$f_i(x) = \begin{cases} a_{1i}, & x = c \\ a_{ki}, & x = b_k,\ 2 \leq k \leq \ell \\ c, & \text{otherwise.} \end{cases}$$

For arbitrary $d_1 \in A_1, d_2 \in A_2, \dots, d_\ell \in A_\ell$ we define operation $g \in O_k^{(1)}$ by

$$g(x) = \begin{cases} d_1, & x \in A_1 \\ d_2, & x \in A_2 \\ & \dots \\ d_\ell, & x \in A_\ell \\ c, & \text{otherwise.} \end{cases}$$

Using the fact that c is a central element, and also total reflexivity and total symmetry of the relation ρ we can prove that $f_1, \dots, f_n, g \in Pol\rho$. Therefore we can define $f' \in \langle Pol\rho \cup \{f\}\rangle_h$ as usual by

$$f' = g\big(f(f_1, \dots, f_n)\big).$$

Then for $(c, b_2, \dots, b_\ell) \in \rho$ we have $\big(f'(c), f'(b_2), \dots, f'(b_\ell)\big) = (d_1, \dots, d_\ell) \notin \rho$ since $(A_1, \dots, A_\ell) \notin \rho_h$, which means that $f' \notin Pol\rho$.

Thus, we have shown that for a central relation ρ hyperclone $hPol\rho$ satisfies the conditions of Theorem 6.3.2, and therefore is a maximal hyperclone.

Example 6.3.11 *Let ρ be a binary central relation on E_3 with the center $C(\rho) = \{2\}$, i.e.,*

$$\rho = \begin{pmatrix} 0 & 1 & 2 & 0 & 2 & 1 & 2 \\ 0 & 1 & 2 & 2 & 0 & 2 & 1 \end{pmatrix}.$$

Let us choose $M = \begin{pmatrix} 0 & 2 \\ 2 & 1 \end{pmatrix} \in \rho^*$ and $f \in H_3^{(2)}$ such that $f(M) = (\{1\}, \{0\})$, which means that $f \notin hPol\rho$. If we choose $c = 2$, $b_2 = 1$ and $d_1 = 1$, $d_2 = 0$, operations $f_1, f_2, g \in O_3^{(1)}$ are defined by

	0	1	2
f_1	2	2	0
f_2	2	1	2
g	0	1	2

Then for operation $f' = g\big(f(f_1, f_2)\big)$ we have $(2, 1) \in \rho$, but $\big(f'(2), f'(1)\big) = (1, 0) \notin \rho$, which implies $f' \notin Pol\rho$.

6.3.4 Hyperclones determined by regular relations

In this chapter we are going to show that $hPol\rho$ is a maximal hyperclone if ρ is a regular relation.

From the definition it is easily deduced that every ℓ-regular relation is totally reflexive and totally symmetric, which will be used in the proof of the following theorem.

Theorem 6.3.12 ([41]) *Let $\rho \subset E_k^\ell$, $\ell \geq 3$, be ℓ-regular relation. Then $hPol\rho$ is a maximal hyperclone on E_k.*

Sketch of a proof. Let f be an n-ary hyperoperation on E_k which is not in $hPol\rho$. Then there exists a matrix

$$M = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{\ell 1} & a_{\ell 2} & \dots & a_{\ell n} \end{pmatrix} \in \rho^*$$

such that $f(M) = (A_1, A_2, \dots, A_\ell) \notin \rho_h$. Again, since ρ is totally reflexive, sets $A_1, \dots, A_\ell$ are pairwise disjoint.

When we can choose distinct elements $b_1, \dots, b_\ell \in E_k$, such that $(b_1, \dots, b_\ell) \in \rho$, we define operations $f_j \in O_k^{(1)}$, $j = 1, \dots, n$ as follows:

$$f_j(x) = \begin{cases} a_{ij}, & x = b_i \\ a_{1j}, & \text{otherwise.} \end{cases}$$

Otherwise we select $(b_1, \dots, b_\ell), (c_1, \dots, c_\ell) \in \rho$ where all pairs (b_1, c_1), $\dots$, (b_ℓ, c_ℓ) are distinct and define operations $f_j \in O_k^{(2)}$, $j = 1, \dots, n$ by:

$$f_j(x, y) = \begin{cases} a_{ij}, & x = b_i \text{ and } y = c_i \\ a_{1j}, & \text{otherwise.} \end{cases}$$

If we choose arbitrary $d_i \in A_i$, $i = 1, \dots, \ell$, then $(d_1, \dots, d_\ell) \notin \rho$, and there also exists an equivalence relation θ^* such that $(d_i, d_j) \notin \theta^*$ for all $i \neq j$. Denote by C_q^* equivalence class of the relation θ^* which includes $q \in E_k$. Since there are no $x \in A_i$ and $y \in A_j$, for $i \neq j$, such that $(x, y) \in \theta^*$, their equivalence classes C_x^* and C_y^* are disjoint. Therefore we can define operation $g \in O_k^{(1)}$ by

$$g(x) = \begin{cases} d_1, & x \in C_{q_1}^*, \text{ for some } q_1 \in A_1 \\ d_2, & x \in C_{q_2}^*, \text{ for some } q_2 \in A_2 \\ & \dots \\ d_\ell, & x \in C_{q_\ell}^*, \text{ for some } q_\ell \in A_\ell \\ d_1, & \text{otherwise.} \end{cases}$$

Total reflexivity and total symmetry of ρ imply that $f_1, \dots, f_n, g \in Pol\rho$. Hence we can define $f' \in \langle Pol\rho \cup \{f\} \rangle_h$ by $f' = g\big(f(f_1, \dots, f_n)\big)$.

Now we have $(b_1, \ldots, b_\ell) \in \rho$ and $\big(f'(b_1), \ldots, f'(b_\ell)\big) = (d_1, \ldots, d_\ell) \notin \rho$ (or $(b_1, \ldots, b_\ell), (c_1, \ldots, c_\ell) \in \rho$ and $\big(f'(b_1, c_1), \ldots, f'(b_\ell, c_\ell)\big) = (d_1, \ldots, d_\ell) \notin \rho$), which implies $f' \notin Pol\rho$.

Consequently, by Theorem 6.3.2, $hPol\rho$, for a regular relation ρ, is a maximal hyperclone.

Example 6.3.13 *Let ρ be a ternary regular relation on E_4, corresponding to the equivalence relation with classes $\{0\}, \{1, 2\}$ and $\{3\}$, i.e.,*

$$\rho = E_4^3 \setminus \begin{pmatrix} 0 & 0 & 1 & 1 & 3 & 3 & 0 & 0 & 2 & 2 & 3 & 3 \\ 1 & 3 & 0 & 3 & 0 & 1 & 2 & 3 & 0 & 3 & 0 & 2 \\ 3 & 1 & 3 & 0 & 1 & 0 & 3 & 2 & 3 & 0 & 2 & 0 \end{pmatrix}.$$

If $M = \begin{pmatrix} 1 & 0 \\ 3 & 1 \\ 2 & 1 \end{pmatrix} \in \rho^*$ *and f is a binary hyperoperation such that* $f(M) = (\{0\}, \{3\}, \{1, 2\})$, *then $f \notin hPol\rho$. If we select $b_1 = 0$, $b_2 = 1$, $b_3 = 2$ and $d_1 = 0$, $d_2 = 3$, $d_3 = 1$, operations $f_1, f_2, g \in O_4^{(1)}$ are given by*

	0	1	2	3
f_1	1	3	2	1
f_2	0	1	1	0
g	0	1	1	3

Define operation $f' = g\big(f(f_1, f_2)\big)$. Then $(0, 1, 2) \in \rho$, and $\big(f'(0), f'(1), f'(2)\big) = (0, 3, 1) \notin \rho$, and therefore $f' \notin Pol\rho$.

6.4 Maximal IS clones

Similarly to the case of hyperclones we can prove that the set of all total operations is a maximal IS clone.

Lemma 6.4.1 ([15, 16]) *For $k \geq 2$, O_k is a maximal IS clone, i.e., for any $f \in I_k \setminus O_k$ we have $\langle O_k \cup \{f\} \rangle_{IS} = I_k$.*

Proof. If $f \in I_k^{(n)} \setminus O_k$, there is at least one n-tuple $(a_1, \ldots, a_n) \in E_k^n$ such that $f(a_1, \ldots, a_n) = k$. It suffices to prove that the statement $\langle O_k \cup \{f\} \rangle_{\text{IS}} \supseteq I_k$

is correct, since the opposite inclusion is straightforward.

Let h be an arbitrary m-ary IS operation from I_k and let us define the mappings $f_1, \ldots, f_n \in O_k^{(m)}$ and $g \in O_k^{(m+1)}$ for all $x_1, \ldots, x_m, x_{m+1} \in E_k$ as follows:

$$\big(f_1(x_1, \ldots, x_m), \ldots, f_n(x_1, \ldots, x_m)\big) = \big(a_1, \ldots, a_n\big)$$

and

$$g(x_1, \ldots, x_{m+1}) = \begin{cases} x_{m+1}, & h(x_1, \ldots, x_m) = k \\ h(x_1, \ldots, x_m), & \text{otherwise.} \end{cases}$$

Now we can prove that

$$h = g\big(e_1^m, \ldots, e_m^m, f(f_1, \ldots, f_n)\big).$$

For $h(x_1, \ldots, x_m) = k$ we have

$$g\big(e_1^m, \ldots, e_m^m, f(f_1, \ldots, f_n)\big)(x_1, \ldots, x_m) = g^+(x_1, \ldots, x_m, k) = k,$$

whereas for $h(x_1, \ldots, x_m) = a \in E_k$,

$$\begin{aligned} g\big(e_1^m, \ldots, e_m^m, f(f_1, \ldots, f_n)\big)(x_1, \ldots, x_m) &= g^+(x_1, \ldots, x_m, k) \\ &= h(x_1, \ldots, x_m) = a. \end{aligned}$$

Therefore, $h \in \langle O_k \cup \{f\} \rangle_{\text{IS}}$. $\square$

Since O_k is finitely generated, it follows directly from the previous lemma that I_k is also finitely generated, and therefore every proper IS clone is a subset of some of the finitely many maximal IS clones.

In what follows we consider four classes of relations on E_k with the property that their weak extensions are equal to their full extensions and state that the corresponding clones of incompletely specified operations are maximal. These results are analogous to the results obtained for maximal hyperclones in [17] and [41], and presented in Section 6.3.

In the following theorem we state that $wPol\rho$ is a maximal clone of incompletely specified operations if the condition (6.3) holds. The proof is analogous to that of Theorem 6.3.2.

Theorem 6.4.2 *Let $Pol\,\rho$ be a maximal clone on E_k such that:*

$$(\forall f \in I_k \setminus wPol\,\rho)\,(\exists f' \in O_k \setminus Pol\,\rho)\;\; f' \in \langle Pol\,\rho \cup \{f\}\rangle_{\mathrm{IS}}. \tag{6.3}$$

Then $wPol\,\rho$ is a maximal clone of incompletely specified operations.

Next lemma asserts that for every totally reflexive relation its weak extension coincides with the full extension.

Lemma 6.4.3 *Let $\ell \geq 2$ and let $\rho \subseteq E_k^\ell$ be a totally reflexive relation. If $(a_1, \ldots, a_\ell) \notin \rho_w$ then $(a_1, \ldots, a_\ell) \in E_k^\ell$.*

Proof. Let us suppose that there exist $(a_1, \ldots, a_\ell) \in E_{k+1}^\ell \setminus (\rho_w \cup E_k^\ell)$. We can assume, without loss of generality, that $a_1 = k$. Choose $b_2, \ldots, b_\ell \in E_k$ such that $(b_2, b_2, \ldots, b_\ell) \sqsubseteq (a_1, a_2, \ldots, a_\ell)$. Then by the total reflexivity of ρ we obtain $(b_2, b_2, \ldots, b_\ell) \in \rho$ and therefore $(a_1, a_2, \ldots, a_\ell) \in \rho_w$. $\square$

Let ρ be a bounded partial order on E_k with the least element $\mathbf{0}$ and the greatest element $\mathbf{1}$. Since ρ is reflexive, according to Lemma 6.4.3, the weak extension ρ_w contains $E_{k+1}^2 \setminus E_k^2$. Moreover, we can conclude that $(b, c) \notin \rho_w$ implies $b, c \in E_k$ and $b \neq c$.

Lemma 6.4.4 *Let $\rho \subseteq E_k^2$ be a bounded partial order and $f \in I_k \setminus wPol\,\rho$. Then there exists $f' \in O_k \setminus Pol\,\rho$ such that $f' \in \langle Pol\,\rho \cup \{f\}\rangle_{\mathrm{IS}}$.*

Proof. Let $f \in I_k \setminus wPol\,\rho$ be of arity n. There is a matrix

$$M = \begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ b_1 & b_2 & \ldots & b_n \end{pmatrix} \in \rho^*$$

such that

$$f(M) = \big(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)\big) = (b, c) \notin \rho_w.$$

Let us define for each $i \in \{1, \ldots, n\}$ the operation $f_i \in O_k^{(1)}$ as follows

$$f_i(x) = \begin{cases} a_i, & x = \mathbf{0} \\ b_i, & \text{otherwise.} \end{cases}$$

We will prove that $f_1, \dots, f_n \in Pol\,\rho$. For $(x, y) \in \rho$ we have

$$\big(f_i(x), f_i(y)\big) = \begin{cases} (a_i, a_i), & \text{for } x = y = \mathbf{0} \\ (a_i, b_i), & \text{for } x = \mathbf{0} \text{ and } y \neq \mathbf{0} \\ (b_i, b_i), & \text{for } x \neq \mathbf{0}, y \neq \mathbf{0}. \end{cases},$$

and therefore $\big(f_i(x), f_i(y)\big) \in \rho$ for every $i \in \{1, \dots, n\}$.

Define $f' \in \langle Pol\,\rho \cup \{f\}\rangle_{\mathrm{IS}}$ as follows

$$f' = f(f_1, \dots, f_n).$$

Thus, for $x \neq \mathbf{0}$ we have

$$\begin{aligned} f'(\mathbf{0}) &= f\big(f_1(\mathbf{0}), \dots, f_n(\mathbf{0})\big) \\ &= f(a_1, \dots, a_n) = b, \\ f'(x) &= f\big(f_1(x), \dots, f_n(x)\big) \\ &= f(b_1, \dots, b_n) = c. \end{aligned}$$

Since $\mathbf{0}$ is the least element, we have $(\mathbf{0}, x) \in \rho$ for each $x \in E_k$. If $x \neq \mathbf{0}$ we get $\big(f'(\mathbf{0}), f'(x)\big) = (b, c) \notin \rho$. Hence, $f' \in O_k \setminus Pol\,\rho$.

□

Nontrivial equivalence relations, central relations and regular relations are all totally reflexive and totally symmetric. In the next two lemmas we will show that they satisfy condition (6.3).

Lemma 6.4.5 *Let $\rho \subseteq E_k^\ell$, $\ell \geq 1$, be a relation with the following properties:*

(i) ρ is totally reflexive and totally symmetric relation,

(ii) there is an ℓ-tuple $(a_1, \dots, a_\ell) \in \rho$ such that $|\{a_1, \dots, a_\ell\}| = \ell$.

If $f \notin wPol\,\rho$ then there is $f' \in \langle Pol\,\rho \cup \{f\}\rangle_{\mathrm{IS}}$ such that $f' \in O_k \setminus Pol\,\rho$.

Proof. Let $f \notin wPol\rho$ be of arity $n \geq 1$. Then there is

$$M = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{\ell 1} & \dots & a_{\ell n} \end{pmatrix} \in \rho^*$$

such that

$$f(M) = \begin{pmatrix} f(a_{11}, \dots, a_{1n}) \\ \dots \\ f(a_{\ell 1}, \dots, a_{\ell n}) \end{pmatrix} = \begin{pmatrix} b_1 \\ \dots \\ b_\ell \end{pmatrix} \notin \rho_w.$$

Since ρ is totally reflexive relation, then Lemma 6.4.3 holds, hence we can deduce that $b_1, \dots, b_\ell$ are distinct elements of E_k.

For distinct elements $a_1, \dots, a_\ell \in E_k$ such that $(a_1, \dots, a_\ell) \in \rho$ we define operations $f_1, \dots, f_n \in O_k^{(1)}$ by

$$f_j(x) = \begin{cases} a_{ij}, & x = a_i \\ a_{1j}, & \text{otherwise} \end{cases}$$

for each $j \in \{1, \dots, n\}$.

We need to prove that $f_1, \dots, f_n \in Pol\,\rho$. Let $(x_1, \dots, x_\ell) \in \rho$. If $x_{k_1} = x_{k_2}$, for some $1 \le k_1 < k_2 \le \ell$, then $f_j(x_{k_1}) = f_j(x_{k_2})$, and by total reflexivity of ρ we get $\big(f_j(x_1), \dots, f_j(x_\ell)\big) \in \rho$. If, on the other hand, all $x_1, \dots, x_\ell$ are distinct, we can distinguish the following cases.

1) If $\{x_1, \dots, x_\ell\} = \{a_1, \dots, a_\ell\}$, using total symmetry of ρ, we may assume that $(x_1, \dots, x_\ell) = (a_1, \dots, a_\ell)$, and then for all $j = 1, \dots, n$ we get

$$\big(f_j(x_1), \dots, f_j(x_\ell)\big) = \big(f_j(a_1), \dots, f_j(a_\ell)\big) = (a_{1j}, \dots, a_{\ell j}) \in \rho;$$

2) There is exactly one $m \in \{1, \dots, \ell\}$ such that $x_m \notin \{a_1, \dots, a_\ell\}$, which means that $f_j(x_m) = a_{1j}$. If a_1 belongs to $\{x_1, \dots, x_\ell\}$, then there are two same coordinates in $\big(f_j(x_1), \dots, f_j(x_\ell)\big)$, which puts it in ρ using total reflexivity. In the case $a_1 \notin \{x_1, \dots, x_\ell\}$ we have

$$\big\{f_j(x_1), \dots, f_j(x_\ell)\big\} = \{a_{1j}, \dots, a_{\ell j}\},$$

which, by total symmetry of ρ, implies $\big(f_j(x_1), \dots, f_j(x_\ell)\big) \in \rho$.

3) If there are $k_1 \neq k_2$ such that $x_{k_1}, x_{k_2} \notin \{a_1, \dots, a_\ell\}$, then $f_j(x_{k_1}) = f_j(x_{k_2}) = a_{1j}$, and thus $\big(f_j(x_1), \dots, f_j(x_\ell)\big) \in \rho$, since ρ is totally reflexive.

Hence, $f_1, \dots, f_n \in Pol\,\rho$.

Let $f' \in \langle Pol\rho \cup \{f\}\rangle_{\mathrm{IS}}$ be defined by

$$f' = f(f_1, \dots, f_n).$$

Then for $i = 1, \dots, \ell$ we obtain

$$f'(a_i) = f\big(f_1(a_i), \dots, f_n(a_i)\big) = f(a_{i1}, \dots, a_{in}) = b_i.$$

Since $(a_1, \dots, a_\ell) \in \rho$ and $\big(f'(a_1), \dots, f'(a_\ell)\big) = (b_1, \dots, b_\ell) \notin \rho$, we deduce that $f' \notin Pol\,\rho$. $\Box$

Property (ii) of Lemma 6.4.5 holds for every nontrivial equivalence relation and for every central relation. However, there are some ℓ-regular relations, e.g., when they correspond to an equality relation, for which we cannot choose distinct elements $a_1, \dots, a_\ell \in E_k$ such that $(a_1, \dots, a_\ell)$ is in the relation.

Lemma 6.4.6 *Let $\rho \in E_k^\ell$ be an ℓ-regular relation such that*

$$\rho = \big\{(x_1, \dots, x_\ell) \,:\, |\{(x_1, \dots, x_\ell)\}| \leq \ell - 1\big\}.$$

Then for every IS operation $f \notin wPol\,\rho$ there exist an operation $f' \in \langle Pol\,\rho \cup \{f\}\rangle_{\mathrm{IS}}$ which is not in $Pol\,\rho$.

Proof. For n-ary IS operation $f \notin wPol\,\rho$ there exists a matrix

$$M = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{\ell 1} & \dots & a_{\ell n} \end{pmatrix} \in \rho^*$$

for which it holds

$$f(M) = \begin{pmatrix} f(a_{11}, \dots, a_{1n}) \\ \dots \\ f(a_{\ell 1}, \dots, a_{\ell n}) \end{pmatrix} = \begin{pmatrix} b_1 \\ \dots \\ b_\ell \end{pmatrix} \notin \rho_w.$$

We choose $(c_1, \dots, c_\ell), (d_1, \dots, d_\ell) \in \rho$ such that the pairs $(c_1, d_1), \dots, (c_\ell, d_\ell)$ are all distinct and define operations $f_1, \dots, f_n \in O_k^{(2)}$ as

$$f_j(x, y) = \begin{cases} a_{ij}, & x = c_i \text{ and } y = d_i \\ a_{1j}, & \text{otherwise} \end{cases}$$

for each $j \in \{1, \ldots, n\}$.

Proof that $f_1, \ldots, f_n \in Pol\,\rho$ is analogue to the proof in Lemma 6.4.5, while here we consider pairs of elements instead of single elements.

Then for $f' = f(f_1, \ldots, f_n) \in \langle Pol\rho \cup \{f\}\rangle_{\mathrm{IS}}$ we have

$$\begin{aligned} f'(c_i, d_i) &= f\big(f_1(c_i, d_i), \ldots, f_n(c_i, d_i)\big) \\ &= f(a_{i1}, \ldots, a_{in}) = b_i, \quad i = 1, \ldots, n. \end{aligned}$$

Given that $(c_1, \ldots, c_\ell), (d_1, \ldots, d_\ell) \in \rho$ and $\big(f'(c_1, d_1), \ldots, f'(c_\ell, d_\ell)\big) = (b_1, \ldots, b_\ell) \notin \rho$, we conclude that $f' \notin Pol\,\rho$. $\square$

Finally, using Theorem 6.4.2 and Lemmas 6.4.4–6.4.6 we obtain the analogous result as in the case of hyperclones.

Corollary 6.4.7 *If ρ is a relation on E_k from one of the following classes:*

(R_1) a bounded partial order,

(R_4) a nontrivial equivalence relation,

(R_5) a central relation, or

(R_6) an ℓ-regular relation,

then $wPol\,\rho$ is a maximal clone of incompletely specified functions.

Example 6.4.8 *From the Corollary 6.4.7 we conclude that the IS clones of the form $wPol\rho_i$ are maximal IS clones on E_3, for the following relations ρ_i, $i \in \{1, 2, \ldots, 16\}$*

$$\begin{array}{lllllllll}
\rho_1 & = & \begin{pmatrix} 012001 \\ 012122 \end{pmatrix} & \rho_6 & = & \begin{pmatrix} 01212 \\ 01221 \end{pmatrix} & \rho_{13} & = & \begin{pmatrix} 0120102 \\ 0121020 \end{pmatrix} \\
\rho_2 & = & \begin{pmatrix} 012112 \\ 012200 \end{pmatrix} & \rho_7 & = & (0) & & & \\
 & & & \rho_8 & = & (1) & \rho_{14} & = & \begin{pmatrix} 0120112 \\ 0121021 \end{pmatrix} \\
\rho_3 & = & \begin{pmatrix} 012220 \\ 012011 \end{pmatrix} & \rho_9 & = & (2) & \rho_{15} & = & \begin{pmatrix} 0120212 \\ 0122021 \end{pmatrix} \\
\rho_4 & = & \begin{pmatrix} 01201 \\ 01210 \end{pmatrix} & \rho_{10} & = & (0\ 1) & & & \\
 & & & \rho_{11} & = & (0\ 2) & \rho_{16} & = & E_3^3 \setminus \begin{pmatrix} 001122 \\ 120201 \\ 212010 \end{pmatrix} \\
\rho_5 & = & \begin{pmatrix} 01202 \\ 01220 \end{pmatrix} & \rho_{12} & = & (1\ 2) & & &
\end{array}$$

Relations ρ_1, ρ_2, ρ_3 are bounded partial orders, ρ_4, ρ_5, ρ_6 are nontrivial equivalence relations, $\rho_7 - \rho_{15}$ are central relations and ρ_{16} is 3-regular relation.

Chapter 7

Concluding remarks

7.1 Applications

Concept of nondeterminism has been an important issue in computer science from its very beginning. Early mentions of nondeterminism appear in the works of McCarthy [43] and Floyd [25] in 1960s and since then a great number of theories and formalisms concerning it have been developed, for instance, denotational models based on power domains, distributed systems, using concurrency and communication, process languages and algebras, algebraic specifications. Extensive list of references can be found in [72].

As we mentioned in Introduction, transition function of a non-deterministic finite automaton may be viewed as a hyperoperation, since the transition from a state can be to more than one (or even none) of the next states for each of the input symbols. Formally, a non-deterministic finite automaton (NFA) is a quintuple $M = (Q, \Sigma, \delta, q_0, F)$, where

- Q is a nonempty finite set of states;
- Σ is a nonempty finite set of input symbols (alphabet);
- $\delta : Q \times \Sigma \to \mathcal{P}(Q)$ is a transition function;
- $q_0 \in Q$ is an initial state;
- $F \subseteq Q$ is a nonempty set of final states.

Although for each non-deterministic finite automaton there exists an equivalent deterministic finite automaton (DFA) which accepts the same language, they do not have the same behaviour. For example, NFA can use an empty string transition, while it is not the case for DFA. Moreover, an NFA is easier to construct and requires less space than DFA, but the time needed to execute an input string in DFA is less than in NFA.

In Figure 7.1 we show equivalent NFA and DFA, i.e., they accept the same language consisting of all stings such that their penultimate symbol is 1.

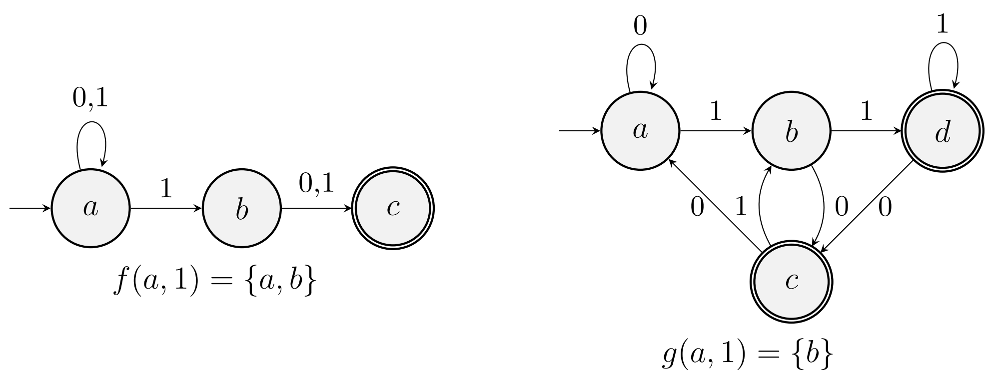

Figure 7.1: NFA and DFA accepting the same language.

We often come across incompletely specified logic functions in theoretical computer science and engineering. They are defined as mappings from D to E_k, with $D \subset E_k^n$, $k \geq 2$, and the input values for which the output values are not specified are referred to as "don't care" conditions. One of the main subjects of research is determining an optimal assignment of unspecified values in order to produce compact representation of such functions.

Example 7.1.1 *Let us consider the expression*

$$E = xy\bar{z} \vee \bar{y}(z \vee \bar{x}\bar{z})$$

(we denote conjunction by $\cdot$ instead of $\wedge$) with the additional condition that $(x, y, z) \notin \{(0, 1, 1), (1, 0, 0)\}$. *Corresponding incompletely specified Boolean function* $f : D \to \{0,1\}$, *where* $D = \{0,1\}^3 \setminus \{(0,1,1),(1,0,0)\}$, *is given by the following truth table*

x	0	0	0	0	1	1	1	1
y	0	0	1	1	0	0	1	1
z	0	1	0	1	0	1	0	1
$f(x,y,z)$	0	1	0	$\times$	$\times$	1	1	1

We can say that on D expression E is equivalent to

$$E' = xy\bar{z} \vee \bar{y}(z \vee \bar{x}\bar{z}) \vee \bar{x}yz \vee x\bar{y}\bar{z}.$$

The minterms $\bar{x}yz$ and $x\bar{y}\bar{z}$ are called "don't care" conditions, since they do not have influence on the value of the expression on D.

If we are looking for a minimal disjunctive normal form of the given expression using Karnaugh maps, "don't care" conditions, represented by $\times$, can be considered either 0 or 1, whatever leads to more minimal solution. (Therefore we may assume the existence of all possible input values, while the output values for some of them are unspecified.) Karnaugh map corresponding to the function f is presented in Figure 7.2. Thus the minimal DNF of E', and

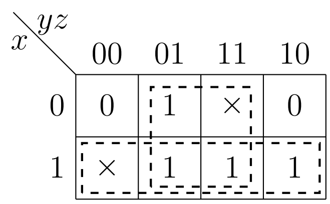

Figure 7.2: Karnaugh map of the function f.

equivalently of E on D, is $x \vee z$.

Incompletely specified operations have found a wide range of applications in computer science, for example in the study of process algebras [3], decision diagrams [46, 76, 69] and switching theory [63, 64].

Certainly the most recent application of clone theory concerns computational complexity of constraint satisfaction problems. Constraint satisfaction problems cover a wide range of both theoretical (satisfiability, coloring, Traveling salesman problem, N-queens problem) and real-life combinatorial problems (scheduling, automated planning, vehicle routing, hardware configuration, networks, bioinformatics) (see [62]).

Given a set of variables, a finite domain and the set of constraints (relations), the objective of the constraint satisfaction problem (CSP) is to determine whether we can assign values from the domain to the variables such that all constraints are satisfied. And we are wondering how complex is this decision.

The general CSP is known to be NP-complete and it is significant to find restrictions that are tractable (decidable in polynomial time).

Considering the fact that the CSPs are defined using relations, we can observe that, by applying the Galois connection (Pol, Inv) between operations and relations, it is possible to correlate CSPs to clones. This was first realised by Jeavons et al. [31, 32], which marked the beginning of the algebraic approach for studying the computational complexity of CSPs rather than just using combinatorial methods. Moreover, Jeavons et al. showed that the complexity of a CSP in many cases depends on a fact that certain functions are contained or not in the corresponding clone.

The most fundamental problem in the area was the famous Dichotomy Conjecture formulated by Feder and Vardi [24].

Conjecture 7.1.2 *For each finite constraint language* Γ*, the problem* $\mathrm{CSP}(\Gamma)$ *is in P or is NP-complete.*

Schaefer in [65] determined all constraint languages Γ over a two-element domain for which $\mathrm{CSP}(\Gamma)$ is in P, and Bulatov in [11] did the same for a three-element domain. Since then a number of partial results have been obtained (see [2] for references) until in 2017 Bulatov in [12] and Zhuk in [77] independently proved the general CSP Dichotomy Conjecture. Using significantly different algorithms they both showed that for a constraint language Γ containing all unary relations, $\mathrm{CSP}(\Gamma)$ is in P iff there exists a weak near-unanimity operation which preserves Γ.

Let us mention that aside from total clones, partial clones have also been used in study of complexity of constraint satisfaction problems [66, 44, 33, 19]. Hence we may anticipate that certain results in this area could be achieved with the help of incompletely specified clones and hyperclones.

7.2 Open problems

There is a number of open problems regarding the lattices of incompletely specified clones and hyperclones that naturally arise from the results presented in this thesis.

In Section 5.2 we presented the result from [20] about the cardinality of intervals

$$\mathcal{I}(C) = \big\{D \subseteq P_2 : D \text{ is partial clone such that } D \cap O_2 = C\big\}.$$

There are only 10 total clones on E_2 whose intervals of this form are finite, and the remaining ones are of continuum cardinality. On a two-element set lattices of IS clones and hyperclones are isomorphic, and it would also be interesting to study, similarly as in the case of partial clones, the sets of all hyperclones whose total part is some total clone on E_2: whether they are intervals and can we describe their position in the lattice $\mathcal{L}_2^h$, what is their cardinality, etc.

Natural way to start this investigation would be to consider maximal clones. For partial clones, it is proved in [1] that if $C \in \{T_0, T_1, S, M\}$, interval $\mathcal{I}(C)$ consists of 6 elements, and $\mathcal{I}(L)$ has cardinality continuum. Useful fact is that by adding all nowhere defined partial operations to an arbitrary partial clone we again obtain a partial clone. Unfortunately, we do not have the same convenience for hyperclones. Namely, if we add to some hyperclone all hyperoperations mapping everything to E_2, the resulting set in general is not the hyperclone, as illustrated by the next example.

Example 7.2.1 *Using hyperoperations from the set $O_2 \cup \langle c_{E_2} \rangle_h$ we can generate a hyperoperation not in this set, as demonstrated in Figure 7.3.*

f	0	1
0	$\{0\}$	$\{1\}$
1	$\{1\}$	$\{1\}$

	g_1	g_2
0	$\{0,1\}$	$\{1\}$
1	$\{0,1\}$	$\{0\}$

	$f(g_1, g_2)$
0	$\{1\}$
1	$\{0,1\}$

Figure 7.3: The composition of the hyperoperations from $O_2 \cup \langle c_{E_2} \rangle_h$.

Although it seams that the lattice of IS clones could easily be embedded into the lattice of hyperclones, the mapping $\eta : I_k \to H_k$ defined by $\eta(f) = f^*$, where

$$f^*(x_1,\dots,x_n)=\begin{cases}\{f(x_1,\dots,x_n)\} & ,f(x_1,\dots,x_n)\in E_k\\ E_k & ,f(x_1,\dots,x_n)=k\end{cases}$$

is not homomorphism because it is not compatible with composition, as it is shown in the following example.

Example 7.2.2 *Let f be a binary and g_1, g_2 unary IS operations on E_3 such that $f(0,2)=f(2,2)=2$, $f(1,2)=1$, $g_1(1)=3$ and $g_2(1)=2$. Than we have*

$$\begin{aligned}\big(f(g_1,g_2)\big)(1)&=f^+\big(g_1(1),g_2(1)\big)=f^+(3,2)\\ &=f(0,2)\sqcap f(1,2)\sqcap f(2,2)=2\sqcap 1\sqcap 2=3,\end{aligned}$$

hence $\big(f(g_1,g_2)\big)^*(1)=\{0,1,2\}$. On the other hand

$$\begin{aligned}\big(f^*(g_1^*,g_2^*)\big)(1)&=(f^*)^{\#}\big(g_1^*(1),g_2^*(1)\big)=(f^*)^{\#}(\{0,1,2\}),\{2\})\\ &=f(0,2)\cup f(1,2)\cup f(2,2)=\{2\}\cup\{1\}\cup\{2\}=\{1,2\}.\end{aligned}$$

Thus $\big(f(g_1,g_2)\big)^*\neq f^*(g_1^*,g_2^*)$.

Consequently, it makes sense to investigate IS clones on their own, for we cannot simply transfer all results from hyperclones.

Since it is proved (in [17] and [41]) that the sets of the form $hPol\,\rho$, i.e., the sets of hyperoperations that weakly preserve relation ρ, are maximal hyperclones whenever ρ is from one of the Rosenberg's classes $(R_1),(R_4),(R_5)$ and (R_6), it is natural to ask if the same holds for relations from classes (R_2) and (R_3). On a two-element set it is not difficult to show that it holds

$$M_3'=hPol\begin{pmatrix}0&1\\1&0\end{pmatrix}\quad\text{and}\quad M_4'=hPol\begin{pmatrix}0&0&0&0&1&1&1&1\\0&0&1&1&0&0&1&1\\0&1&0&1&0&1&0&1\\0&1&1&0&1&0&0&1\end{pmatrix},$$

(M_3' and M_4' are maximal hyperclones from Theorem 5.3.1), which gives us hope that it might be the case.

For every Rosenberg's relation ρ the set $hPol\,\rho$ is a maximal hyperclone if for each hyperoperation f not in $hPol\,\rho$ there exists an operation f' not in $Pol\,\rho$ which is generated by f and some operations from $Pol\,\rho$ (Theorem 6.3.2). This sufficient condition is used to prove the theorems concerning classes $(R_1), (R_4), (R_5)$ and (R_6), in a way that the operation f' is effectively constructed. What makes this construction pretty much straightforward in the case of nontrivial equivalence relations, central relations and regular relations, especially the proof that all the auxiliary operations are in $Pol\,\rho$, is the fact that all these relations are both totally reflexive and totally symmetric. On the other hand, bounded partial orders are just reflexive, and the absence of symmetry results in more complicated definition of the auxiliary operation $g^{\rho}_{b,c}$ and hence quite cumbersome proof of the fact that it preserves a given order relation ρ.

Nevertheless, in general, permutational relations and affine relations are neither totally reflexive nor totally symmetric (with the exception of permutational relations corresponding to permutations on E_k with $k/2$ transpositions, which are symmetric). This inconvenience still poses an insurmountable obstacle if we try to use the same technique as in the solved cases. It seams that this problem requires a different approach.

As we saw in Section 6.4, proofs that $wPol\,\rho$ is a maximal IS clone if ρ is from one of the classes $(R_1), (R_4), (R_5)$ and (R_6) are much simpler than in the case of hyperclones. Although we are still missing total reflexivity and total symmetry for permutational and affine relations, their weak extension is now equal to their full extension, which should be a great asset in overcoming existing difficulties.

After completing previous tasks, another challenging problem would be to describe all maximal hyperclones and maximal IS clones on a three-element set, or even work towards the general completeness criteria for lattices $\mathcal{L}_k^h$ and $\mathcal{L}_k^{\mathrm{IS}}$.

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Овај Образац чини саставни део докторске дисертације, односно докторског уметничког пројекта који се брани на Универзитету у Новом Саду. Попуњен Образац укоричити иза текста докторске дисертације, односно докторског уметничког пројекта.

План третмана података

Назив пројекта/истраживања
Clones of Nondeterministic Operations / Клонови недетерминистичких операција
Назив институције/институција у оквиру којих се спроводи истраживање
Факултет техничких наука, Универзитет у Новом Саду
Назив програма у оквиру ког се реализује истраживање
Математика у техници (докторске студије)
1. Опис података

1.1 Врста студије

Укратко описати тип студије у оквиру које се подаци прикупљају

Докторска дисертација

1.2 Врсте података

а) квантитативни

б) **квалитативни**

1.3. Начин прикупљања података

а) анкете, упитници, тестови

б) клиничке процене, медицински записи, електронски здравствени записи

в) генотипови: навести врсту ________________________________

г) административни подаци: навести врсту ________________________

д) узорци ткива: навести врсту___________________________________

ђ) снимци, фотографије: навести врсту______________________________

е) **текст:** Актуелна литература у области истраживања

ж) мапа, навести врсту ____

з) остало: описати ____

1.3 Формат података, употребљене скале, количина података

1.3.1 Употребљени софтвер и формат датотеке:

а) Excel фајл, датотека ___________________

b) SPSS фајл, датотека ___________________

c) PDF фајл, датотека ____________________

d) Текст фајл, датотека ___________________

e) JPG фајл, датотека ____________________

f) Остало, датотека _____________________

1.3.2. Број записа (код квантитативних података)

а) број варијабли ____

б) број мерења (испитаника, процена, снимака и сл.) _______________________

1.3.3. Поновљена мерења

а) да

б) не

Уколико је одговор да, одговорити на следећа питања:

а) временски размак између поновљених мера је _______________________________

б) варијабле које се више пута мере односе се на __________________________________

в) нове верзије фајлова који садрже поновљена мерења су именоване као _____________

Напомене: ____

Да ли формати и софтвер омогућавају дељење и дугорочну валидност података?

а) Да

б) Не

Ако је одговор не, образложити ____

____

2. Прикупљање података

2.1 Методологија за прикупљање/генерисање података

2.1.1. У оквиру ког истраживачког нацрта су подаци прикупљени?

а) експеримент, навести тип ____

б) корелационо истраживање, навести тип ____

ц) анализа текста, навести тип ____

д) остало, навести шта ____

2.1.2 *Навести врсте мерних инструмената или стандарде података специфичних за одређену научну дисциплину (ако постоје).*

____

2.2 Квалитет података и стандарди

2.2.1. Третман недостајућих података

а) Да ли матрица садржи недостајуће податке? Да Не

Ако је одговор да, одговорити на следећа питања:

а) Колики је број недостајућих података? ___________________________

б) Да ли се кориснику матрице препоручује замена недостајућих података? Да Не

в) Ако је одговор да, навести сугестије за третман замене недостајућих података

____

2.2.2. На који начин је контролисан квалитет података? Описати

____

2.2.3. На који начин је извршена контрола уноса података у матрицу?

____

3. Третман података и пратећа документација

3.1. Третман и чување података

3.1.1. *Подаци ће бити депоновани у* ____________________________________ *репозиторијум.*

3.1.2. *URL адреса* ____

3.1.3. *DOI* ____

3.1.4. *Да ли ће подаци бити у отвореном приступу?*

а) Да

б) Да, али после ембарга који ће трајати до ____________________________________

в) Не

Ако је одговор не, навести разлог ____

3.1.5. *Подаци неће бити депоновани у репозиторијум, али ће бити чувани.*

Образложење

____

3.2 Метаподаци и документација података

3.2.1. Који стандард за метаподатке ће бити примењен? __________________________________

3.2.1. Навести метаподатке на основу којих су подаци депоновани у репозиторијум.

___

Ако је потребно, навести методе које се користе за преузимање података, аналитичке и процедуралне информације, њихово кодирање, детаљне описе варијабли, записа итд.

___

3.3 Стратегија и стандарди за чување података

3.3.1. До ког периода ће подаци бити чувани у репозиторијуму? ___

3.3.2. Да ли ће подаци бити депоновани под шифром? Да Не

3.3.3. Да ли ће шифра бити доступна одређеном кругу истраживача? Да Не

3.3.4. Да ли се подаци морају уклонити из отвореног приступа после извесног времена?

Да Не

Образложити

___

4. Безбедност података и заштита поверљивих информација

Овај одељак МОРА бити попуњен ако ваши подаци укључују личне податке који се односе на учеснике у истраживању. За друга истраживања треба такође размотрити заштиту и сигурност података.

4.1 Формални стандарди за сигурност информација/података

Истраживачи који спроводе испитивања с људима морају да се придржавају Закона о заштити података о личности (*https://www.paragraf.rs/propisi/zakon_o_zastiti_podataka_o_licnosti.html*) и одговарајућег институционалног кодекса о академском интегритету.

4.1.2. Да ли је истраживање одобрено од стране етичке комисије? Да **Не**

Ако је одговор Да, навести датум и назив етичке комисије која је одобрила истраживање

___

4.1.2. Да ли подаци укључују личне податке учесника у истраживању? Да **Не**

Ако је одговор да, наведите на који начин сте осигурали поверљивост и сигурност информација везаних за испитанике:

а) Подаци нису у отвореном приступу

б) Подаци су анонимизирани

ц) Остало, навести шта ___

5. Доступност података

5.1. *Подаци ће бити*

а) јавно доступни

б) доступни само уском кругу истраживача у одређеној научној области

ц) затворени

Ако су подаци доступни само уском кругу истраживача, навести под којим условима могу да их користе:

____

Ако су подаци доступни само уском кругу истраживача, навести на који начин могу приступити подацима:

____

5.4. *Навести лиценцу под којом ће прикупљени подаци бити архивирани.*

____

6. Улоге и одговорност

6.1. *Навести име и презиме и мејл адресу власника (аутора) података*

Јелена Чолић Оравец, e-mail: **j_colic@uns.ac.rs**

6.2. *Навести име и презиме и мејл адресу особе која одржава матрицу с подацима*

____

6.3. *Навести име и презиме и мејл адресу особе која омогућује приступ подацима другим истраживачима*

____

PERSONAL INFORMATION

Name: Jelena Čolić Oravec

Date, place and country of birth: 30.03.1980, Bačka Palanka, Serbia

Marrital status: Married, 2 children

Home address: Partijarha Rajačića 56, 21131 Petrovaradin

Office address: Trg Dositeja Obradovića 6, 21000 Novi Sad

e-mail: j_colic@uns.ac.rs

Personal site: https://sites.google.com/site/jelenacolicoravec/

ACADEMIC RECORD

2012 – 2022	PhD studies in Applied Mathematics, Faculty of Technical Sciences, University of Novi Sad
2010 – 2012	M.Sc. degree in Theoretical Mathematics, Faculty of Sciences, University of Novi Sad, GPA: 9.95/10
1999 – 2003	B.Sc. degree in Theoretical Mathematics, Faculty of Sciences, University of Novi Sad, GPA: 9.97/10

WORK POSITIONS

Since 2010	Teaching assistant at Faculty of Technical Sciences, University of Novi Sad
2010	Mathematics teacher at Gymnasium “Svetozar Miletić”, Srbobran
2008 – 2010	Research assistant at Faculty of Sciences, University of Novi Sad
2007 – 2015	Geometry teacher at Gymnasium “Jovan Jovanović Zmaj”, Novi Sad
2007 – 2008	Teaching assistant at Higher School of Professional Buisness Studies, Novi Sad,
2006 – 2007	Teaching assistant at Faculty of Technology, University of Novi Sad
2004 – 2006	Mathematics teacher at Primary school “Zdravko Čelar”, Čelarevo

PUBLICATIONS

UNIVERSITY TEXTBOOKS

2018	Jelena Čolić Oravec, *Zbirka rešenih ispitnih zadataka iz Algebre*, FTN izdavaštvo, Novi Sad

JOURNAL PAPERS

2015	Jelena Čolić Oravec, Hajime Machida, Jovanka Pantović, Gradimir Vojvodić: *From Clones to Hyperclones*, Collection of Papers 18(26): Selected Topics in Logic in Computer Science, pp. 111-144
2015	Jelena Čolić Oravec, Hajime Machida, Jovanka Pantović: *Upward Saturated Hyperclones*, Journal of Multiple-Valued Logic and Soft Computing, 24(1-4): pp. 189-201
2014	Jelena Čolić Oravec, Hajime Machida, Jovanka Pantović: *One-point Extension of the Algebra of Incompletely Specified Operations*, Journal of Multiple-Valued Logic and Soft Computing, 22(1-2): pp. 79-94

CONFERENCE PAPERS

2013	Jelena Čolić Oravec, Hajime Machida, Jovanka Pantović: *On Hyper Co-clones*, Proceedings of ISMVL 2013: pp. 182-185
2012	Jelena Čolić Oravec, Hajime Machida, Jovanka Pantović: *Clones of Incompletely Specified Operations*, Proceedings of ISMVL 2012: pp. 256-261
2011	Jelena Čolić Oravec, Hajime Machida, Jovanka Pantović: *Maximal Hyperclones Determined by Monotone Operations*, Proceedings of ISMVL 2011: pp. 160-163

RESEARCH

INTERESTS

Discrete mathematics, clone theory, models of nondeterministic processes, reversible computation

PROJECTS

2015 – 2019	Cost Action IC1405: Reversible Computation - Extending Horizons of Computing
2013 – 2015	DART: Dynamically and Autonomously Reconfigurable Types, bilateral project between Italy and Serbia
2012 – 2016	Cost Action IC1201: Behavioural Types for Reliable Large-Scale Software Systems (BETTY)
2011 – 2020	Representations of logical structures and formal languages and their application in computing, Ministry of Education and Science, Project ON 174026
2008 – 2010	Forcing, Model Theory and Set-Theoretic Topology 2, Ministry of Science and Environmental Protection, Project No 144001

ACTIVITIES

SUMMER AND WINTER SCHOOLS

September 6 – 12, 2014	SSAOS 2014 – Summer School on General Algebra and Ordered Sets, Stara Lesna, Slovakia
July 3 – 5, 2013	Third International SAT/SMT Summer School 2013, Espoo, Finland
February 6 – 10, 2012	Winter School on Verification, Vienna, Austria
November 21 – December 3, 2004	DAAD Intensive Course on Neural Networks, Sofia, Bulgaria

CONFERENCES AND WORKSHOPS

June 4 – 6, 2021	AAA101 – 101st Workshop on General Algebra, online event
May 25 – 27, 2021	ISMVL 2021 – 51st IEEE International Symposium on Multiple-Valued Logic, online event
June 15 – 18, 2017	AAA94+NSAC 2017 – 94th Workshop on General Algebra & 5th Novi Sad Algebraic Conference, Novi Sad, Serbia
May 22 – 24, 2017	ISMVL 2017 – 47th IEEE International Symposium on Multiple-Valued Logic, Novi Sad, Serbia
May 23 – 26, 2016	TYPES 2016 – 22nd International Conference on Types for Proofs and Programs, Novi Sad, Serbia
September 21 – 25, 2015	LAP 2015 – Logic and Applications, Dubrovnik, Croatia
June 5 – 7, 2015	AAA90 – 90th Workshop on General Algebra, Novi Sad, Serbia
May 18 – 20, 2015	ISMVL 2015 – 45th IEEE International Symposium on Multiple-Valued Logic, Waterloo, Ontario, Canada
June 2014	MACORS 2014 – 4th Mathematical Conference of the Republic of Srpska, Trebinje, BiH
June 5 – 9, 2013	NSAC 2013 – 4th Novi Sad Algebraic Conference, Novi Sad, Serbia
March 30, 2013	Workshop on Progress in Decision Procedures: From Formalizations to Applications, Belgrade, Serbia
March 18 – 22, 2013	GAMM 2013 – 84th Annual Meeting of the International Association of Applied Mathematics and Mechanics, Novi Sad, Serbia
June 21 – 25, 2012	Conference on Universal Algebra and Lattice Theory, Szeged, Hungary

March 15 – 18, 2012	AAA83 – 83rd Workshop on General Algebra, Novi Sad, Serbia
February 3 – 4, 2012	5th Workshop on Formal Theorem Proving and Applications, Belgrade, Serbia
May 29 – June 3, 2011	RDP 2011 – Federated Conference on Rewriting, Deduction and Programming, Novi Sad, Serbia
August 17 – 21, 2009	NSAC 2009 – 3rd Novi Sad Algebraic Conference, Novi Sad, Serbia
December, 2007	Dani logike, Novi Sad, Serbia