# Dinamika
# Dinamika krutog tela

Mehanika

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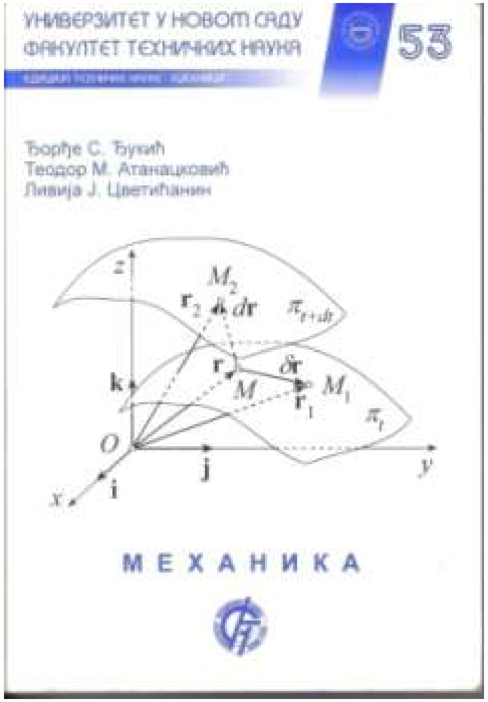
# Literatura

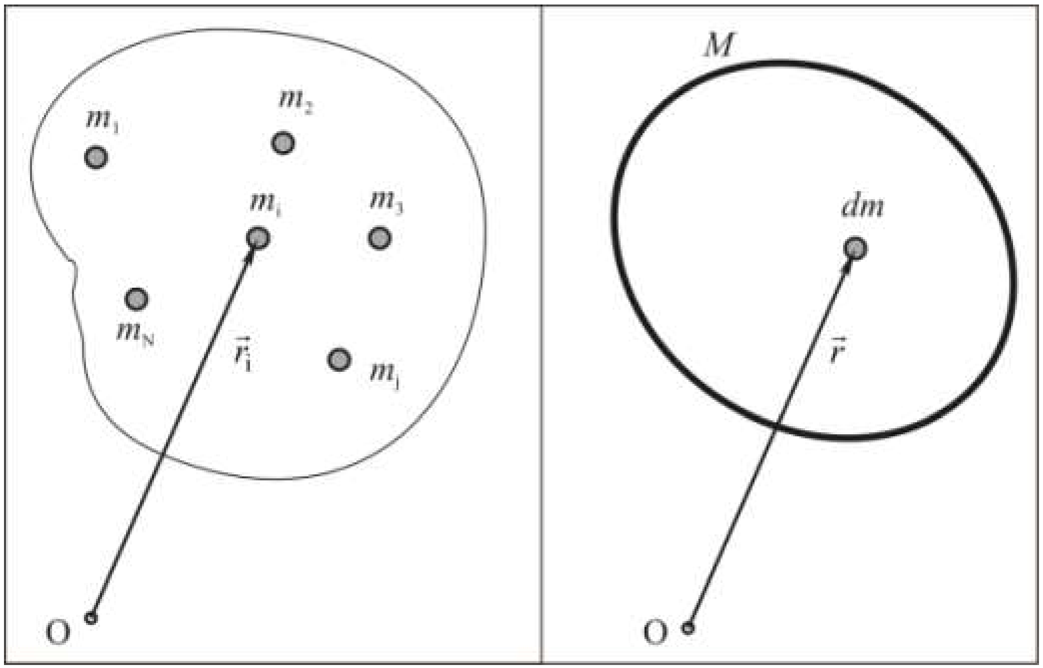
- Đorđe S. Đukić, Teodor M. Atanacković, Livija J. Cvetićanin: **Mehanika**, Fakultet tehničkih nauka u Novom Sadu, Novi Sad, 2003.



# Šta ćemo naučiti?

- Translatorno kretanje
- Obrtanje oko nepokretne ose
- Ravansko kretanje

# Sistem tačaka – kruto telo




$$M = \sum_{i=1}^{N} m_i \to M = \int_V dm$$

$$M\vec{r}_C = \sum_{i=1}^{N} m_i \vec{r}_i \to M\vec{r}_C = \int_V \vec{r}\, dm$$

$$\vec{L}_O = \sum_{i=1}^{N} \vec{L}_{Oi} = \sum_{i=1}^{N} \vec{r}_i \times m_i \vec{v}_i \to$$

$$\vec{L}_O = \int_V d\vec{L}_O = \int_V \vec{r} \times \vec{v}\, dm$$

$$E_K = \sum_{i=1}^{N} E_{Ki} = \sum_{i=1}^{N} \frac{1}{2} m_i {v_i}^2 \to$$

$$E_K = \int_V \frac{1}{2} v^2 dm$$

$$a = \sum_{i=1}^{N} m_i b_i \to a = \int_V b\, dm$$

# Translatorno kretanje

- Diferencijalna jednačina kretanja

$$\boxed{M\vec{a}_C = \vec{F}_g^{\,s}}$$

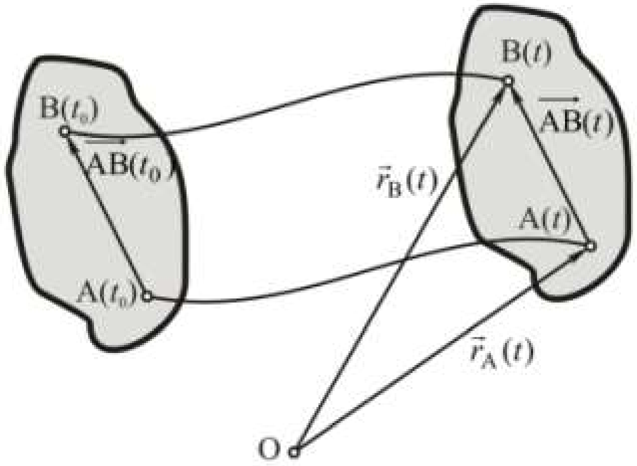
- Zakon o promeni momenta količine kretanja (uslov translatornog kretanja)

$$\vec{v}_r = \vec{v}_i - \vec{v}_C = 0$$

$$d\vec{L}_C = \vec{\rho} \times dm\vec{v}_r = 0$$

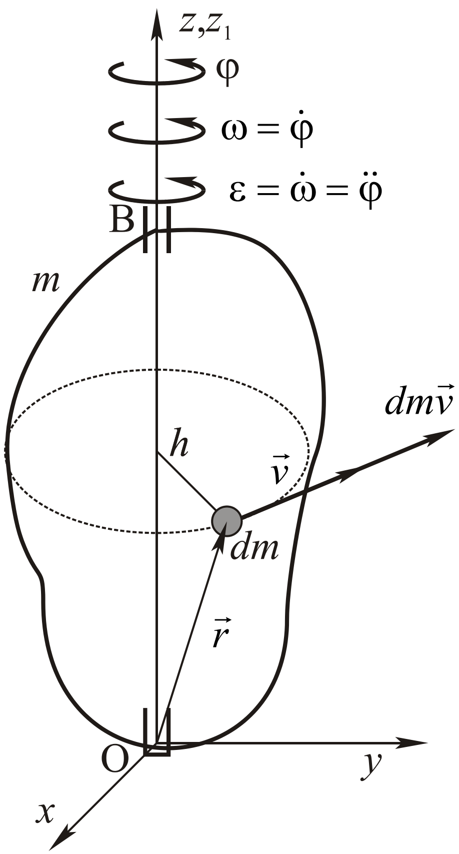
$$\vec{L}_C \equiv 0 \to \dot{\vec{L}}_C \equiv 0$$

$$\boxed{\dot{\vec{L}}_C = \overrightarrow{\mathrm{M}}_{gC}^{\,s} \to \overrightarrow{\mathrm{M}}_{gC}^{\,s} = 0}$$



# Obrtanje oko nepokretne ose

- Diferencijalna jednačina kretanja




$$\frac{d\vec{L}_O}{dt} = \vec{M}_O^s \rightarrow \boxed{\frac{dL_z}{dt} = \sum M_z^s}$$

$$dL_z = h\,dm\,v = h^2\omega\,dm \rightarrow L_z = \omega \int_B h^2\,dm$$

$$\boxed{J_z = \int_B h^2\,dm} \qquad \boxed{L_z = J_z\omega = J_z\dot{\varphi}} \qquad \boxed{\frac{dL_z}{dt} = J_z\dot{\omega} = J_z\varepsilon = J_z\ddot{\varphi}}$$

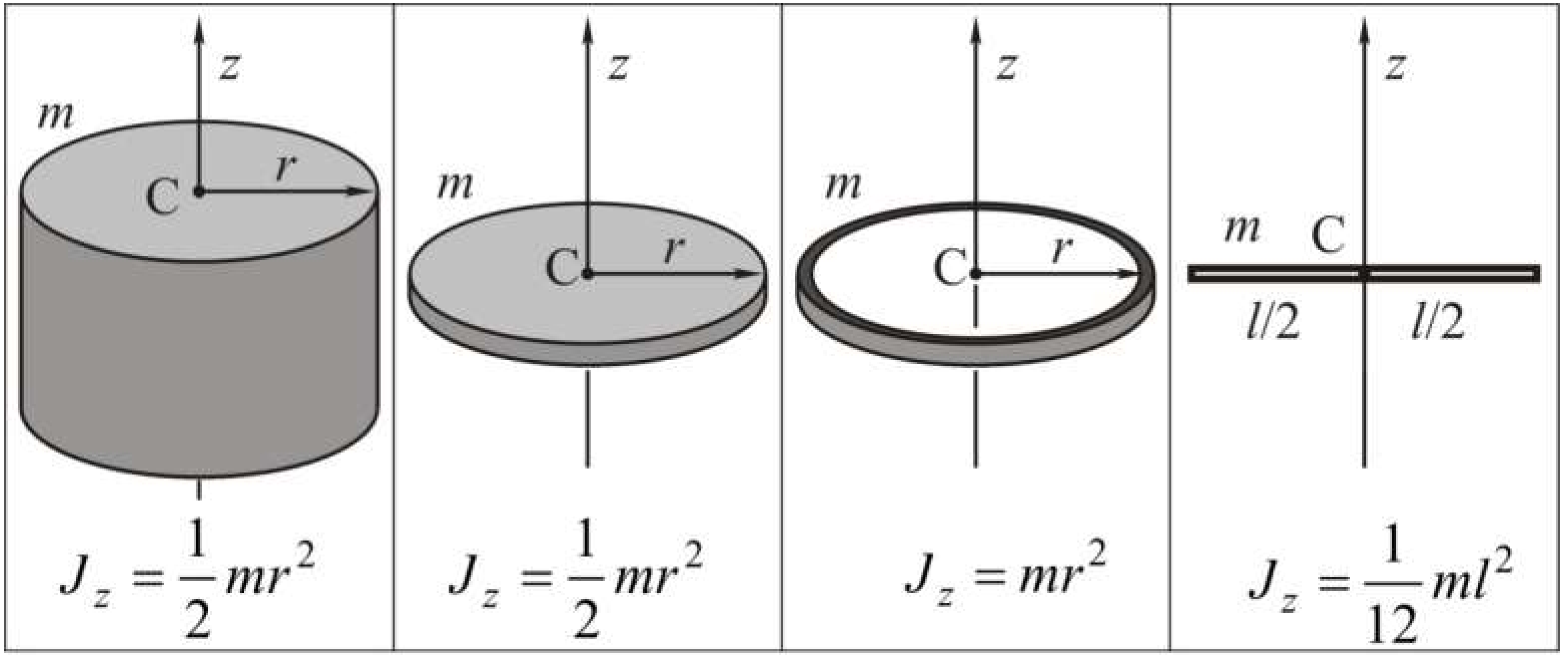
$$J_z\varepsilon = \sum M_z^s \qquad \boxed{J_z\ddot{\varphi} = \sum M_z^s}$$

- Kinetička energija

$$\boxed{E_K = \frac{1}{2} J_z \omega^2}$$

# Obrtanje oko nepokretne ose

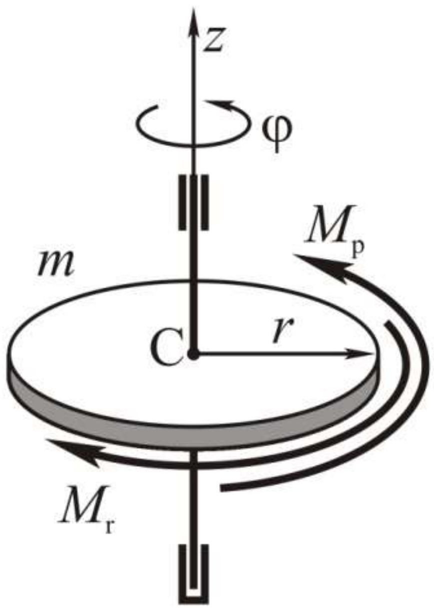
- Moment inercije - primeri



# Obrtanje oko nepokretne ose

- **Primer**

Disk, mase *m* i poluprečnika *r*, obrće se oko nepokretne ose. Na njega deluje pogonski moment koji je funkcija ugaone brzine – $M_p = M_0\left(1 - \frac{\dot{\varphi}}{\Omega}\right)$ i konstantan radni moment $M_r$. Odrediti zakon promene ugaone brzine diska, ako onkreće iz stanja mirovanja.



$$J_z\ddot{\varphi} = M_p - M_r; \quad J_z = \frac{mr^2}{2}$$

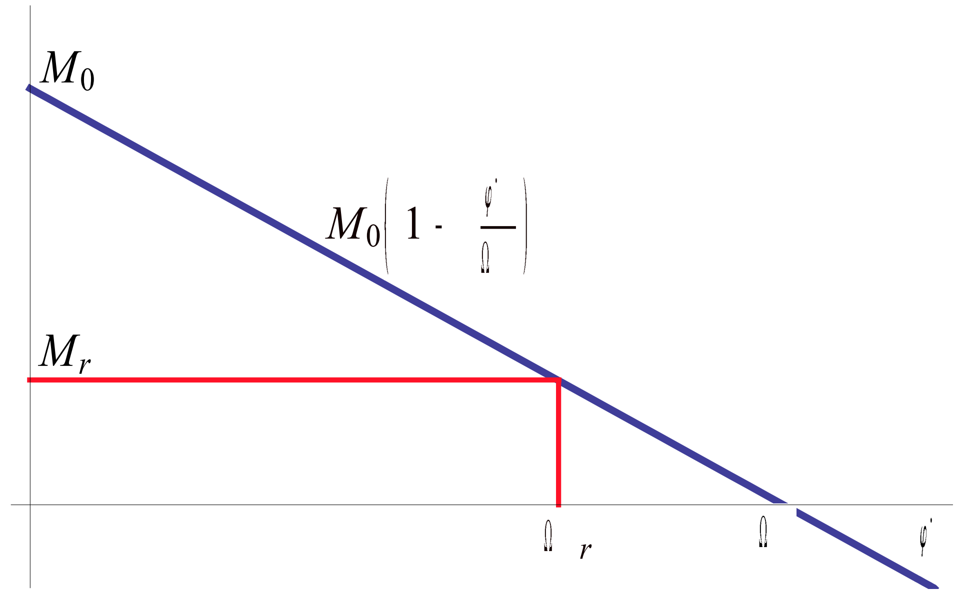
$$J_z\frac{d\dot{\varphi}}{dt} = M_0\left(1-\frac{\dot{\varphi}}{\Omega}\right) - M_r$$

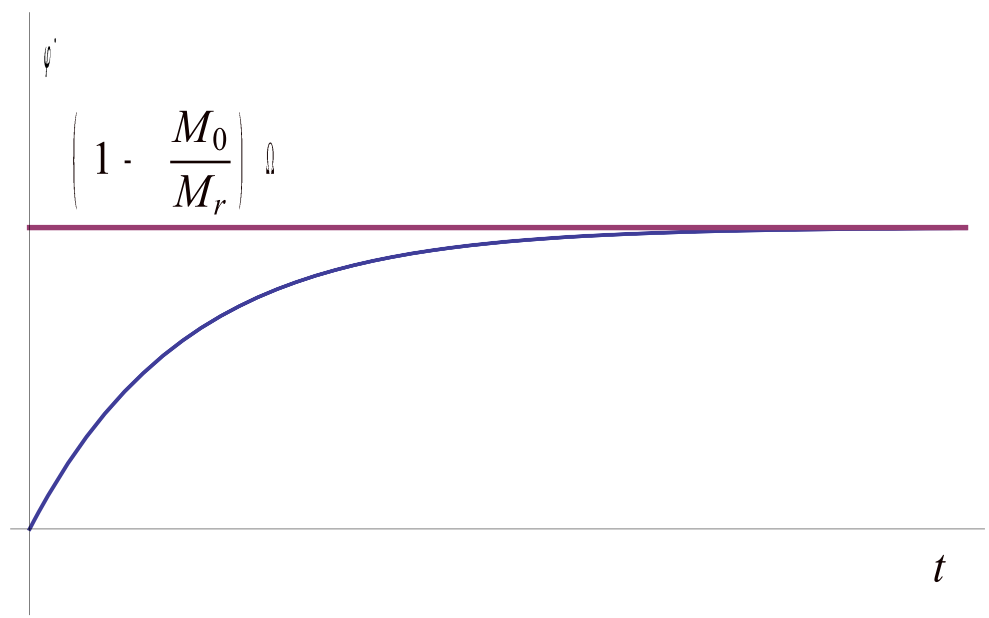
$$\int\limits_{\dot{\varphi}(0)=0}^{\dot{\varphi}} \frac{J_z}{M_0\left(1-\frac{\dot{\varphi}}{\Omega}\right)-M_r}\, d\dot{\varphi} = \int\limits_0^t dt$$

$$-\frac{J_z\Omega}{M_0}\ln\left(\frac{M_0\left(1-\frac{\dot{\varphi}}{\Omega}\right)-M_r}{M_0-M_r}\right) = t$$

$$\frac{M_0\left(1-\frac{\dot{\varphi}}{\Omega}\right)-M_r}{M_0-M_r} = e^{-\frac{M_0}{J_z\Omega}t}$$

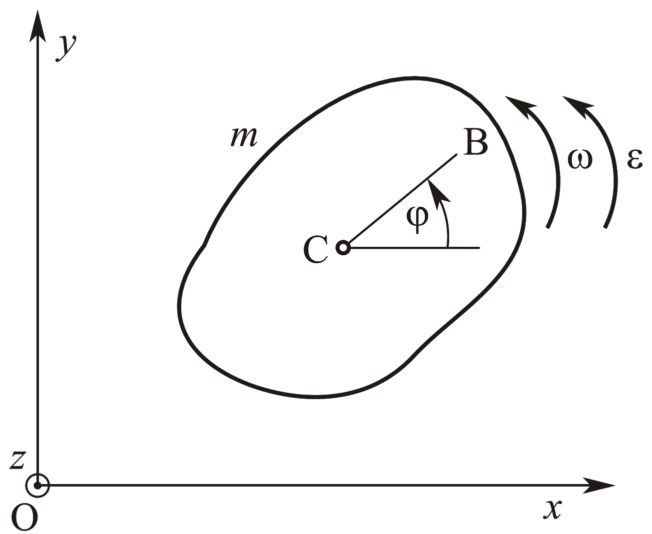
$$\boxed{\dot{\varphi} = \Omega\left(1-\frac{M_r}{M_0}\right) - \Omega\left(1-\frac{M_r}{M_0}\right)e^{-\frac{M_0}{J_z\Omega}t}}$$



# Ravansko kretanje

- Diferencijalne jednačina kretanja




$$m\vec{a}_C = \vec{F}_g^s \rightarrow \begin{cases} m\ddot{x}_C = \sum F_{ix}^s & (1) \\ m\ddot{y}_C = \sum F_{iy}^s & (2) \end{cases}$$

$$\frac{d\vec{L}_C}{dt} = \vec{M}_C^s \rightarrow J_C\ddot{\varphi} = \sum M_C^s \qquad (3)$$

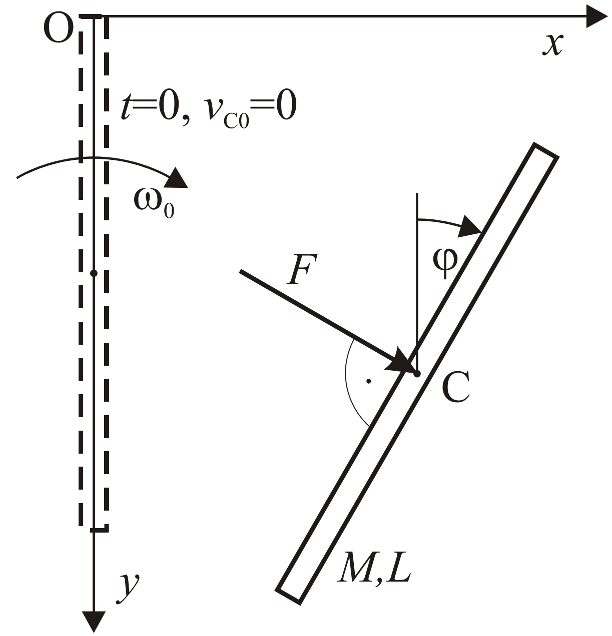
Kinetička energija

$$E_K = \frac{1}{2}mv_C^2 + \frac{1}{2}J_C\omega^2$$

# Ravansko kretanje

- Primer

Štap, mase $M$ i dužine $L$, kreće se po horizontalnom glatkom stolu, pri čemu na njega deluje sila konstantnog intenziteta $F$, čija napadna linija stalno prolazi kroz centar štapa i upravna je na njega. Odrediti kretanje štapa, ($x_C(t)$, $y_C(t)$, φ(t)), ako se on u početnom trenutku nalazio na $y$ osi i ako mu je brzina centra bila jednaka nuli, a ugaona brzina $\omega_0$.

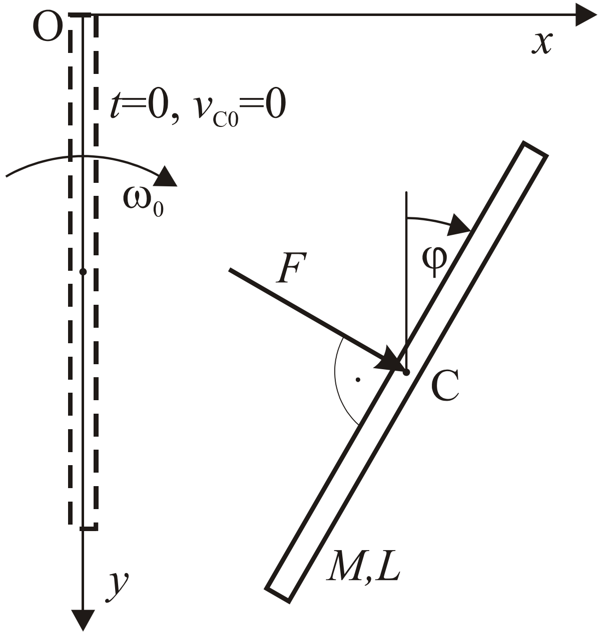


# Ravansko kretanje

- Primer

$$M\vec{a}_C = \vec{F}_g^s \rightarrow \begin{cases} M\ddot{x}_C = F\cos\varphi & (1) \\ M\ddot{y}_C = F\sin\varphi & (2) \end{cases}$$

$$J_C\ddot{\varphi} = \sum M_C^s \rightarrow \quad J_C\ddot{\varphi} = 0 \quad (3)$$



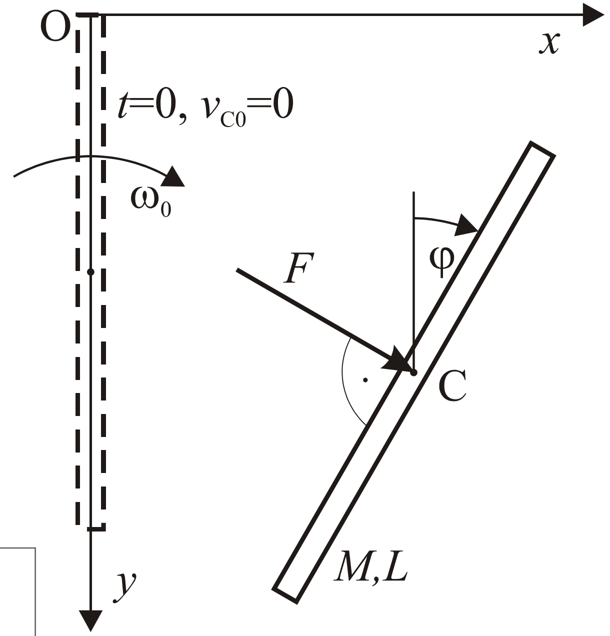
# Ravansko kretanje

- Primer

$$M\vec{a}_C = \vec{F}_g^s \rightarrow \begin{cases} M\ddot{x}_C = F\cos\varphi & (1) \\ M\ddot{y}_C = F\sin\varphi & (2) \end{cases}$$

$$J_C\ddot{\varphi} = \sum M_C^s \rightarrow \quad J_C\ddot{\varphi} = 0 \quad (3)$$

$$(3) \rightarrow \ddot{\varphi} = 0 \rightarrow \dot{\varphi} = C = \omega_0 \rightarrow \varphi = \omega_0 t$$




$$(1) \rightarrow M\ddot{x}_C = F\cos\omega_0 t \rightarrow$$

$$\int_0^{\dot{x}_C} d\dot{x}_C = \frac{F}{M}\int_0^t \cos\omega_0 t\, dt \rightarrow \dot{x}_C = \frac{F}{M\omega_0}\sin\omega_0 t \rightarrow$$

$$\int_0^{x_C} dx_C = \frac{F}{M\omega_0}\int_0^t \sin\omega_0 t\, dt \rightarrow x_C = -\frac{F}{M{\omega_0}^2}(\cos\omega_0 t - 1)$$

# Šta smo naučili?

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- Obrtanje oko nepokretne ose
- Ravansko kretanje