Dinamika
Dinamika krutog tela,…

Kinematika i dinamika

Miodrag Zuković

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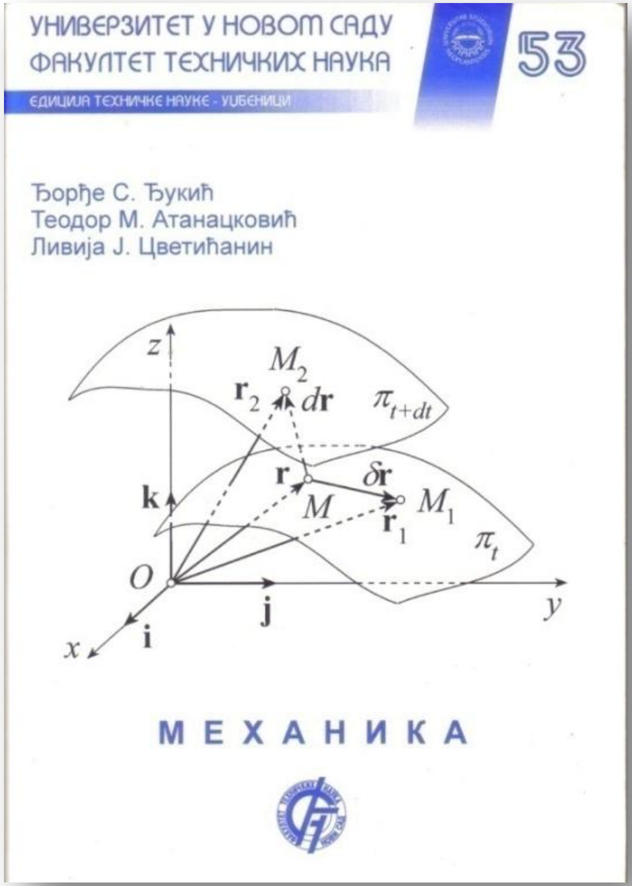
Literatura

- Đorđe S. Đukić, Teodor M. Atanacković, Livija J. Cvetićanin: Mehanika, Fakultet tehničkih nauka u Novom Sadu, Novi Sad, 2003.



Šta ćemo naučiti?

37. **Translatorno kretanje tela.**
38. **Moment inercije tela. Štajnerova teorema.**
39. **Diferencijalna jednačina obrtanja oko nepomične ose. Kinetička energija obrtanja.**
40. **Diferencijalne jednačine ravanskog kretanja tela. Kinetička energija.**
41. **Ravansko kretanje sistema krutih tela.**
42. **Osnove teorije udara.**

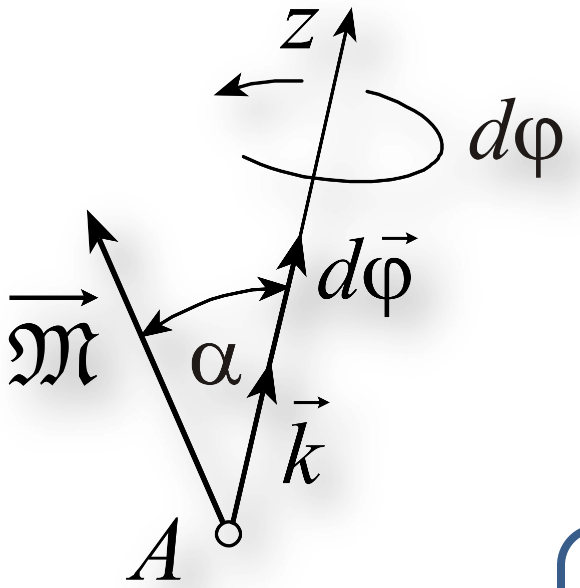
Rad unutrašnjih sila krutog tela

- Rad unutrašnjih sila krutog tela u zakonu promene kinetičke energije sistema ili u zakonu o održanju totalne mehaničke energije jednak je nuli.

- Pošto se tako u svim opštim zakonima dinamike pri proučavanju kretanja krutog tela unutrašnje sile ne pojavljuju, dalje se koristi oznaka:

$$\boxed{\vec{F}_i^{(s)} \rightarrow \vec{F}_i}$$

Rad sprega i momenta sile



$$\boxed{dA^{\overrightarrow{\mathfrak{M}}} = \overrightarrow{\mathfrak{M}} \cdot d\vec{\varphi}}$$

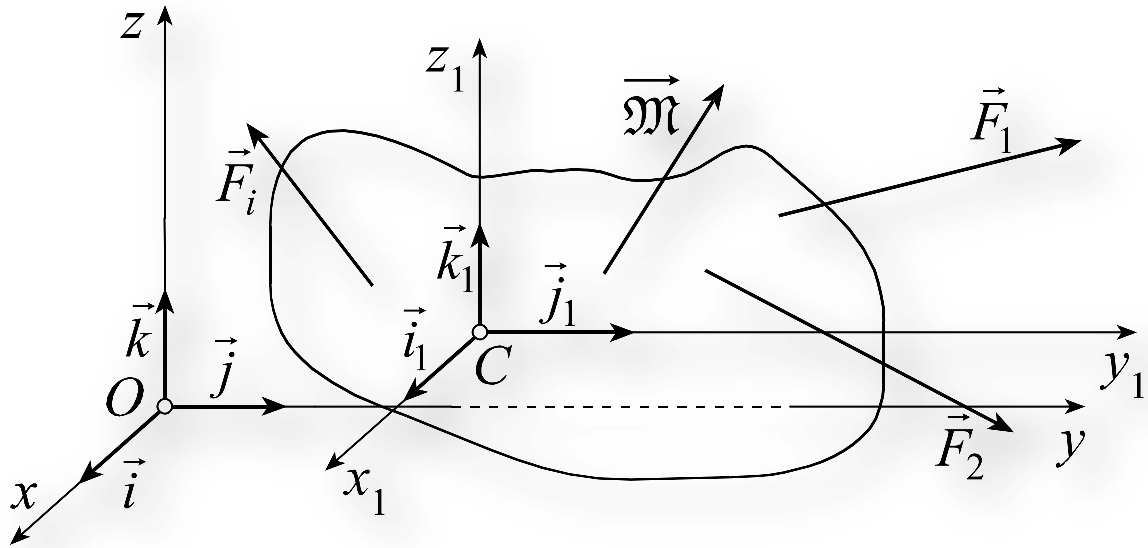
$$\boxed{dA^{\overrightarrow{\mathfrak{M}}} = d\varphi \mathfrak{M} \cos\alpha = \mathfrak{M}_z d\varphi}$$

37. Translatorno kretanje tela

Diferencijalne jednačine kretanja:

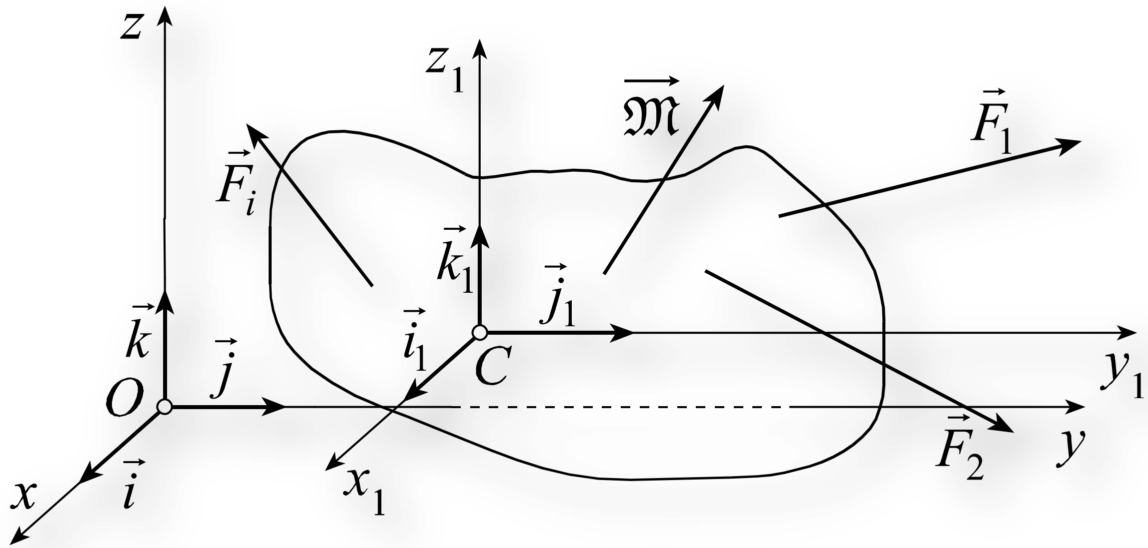
$$M\,\vec{a}_C = \vec{F}_g^{\,s}$$

$$M\,\vec{a}_C = \sum_{i=1}^{N} \vec{F}_i^{\,s}$$



$$M\,\ddot{x}_C = \sum_{i=1}^{N} F_{ix}^{s}, \quad M\,\ddot{y}_C = \sum_{i=1}^{N} F_{iy}^{s}, \quad M\,\ddot{z}_C = \sum_{i=1}^{N} F_{iz}^{s}$$

Uslov(i) translatornog kretanja:



$$\dot{\vec{L}}_C = \vec{M}_{gC}^{s}$$

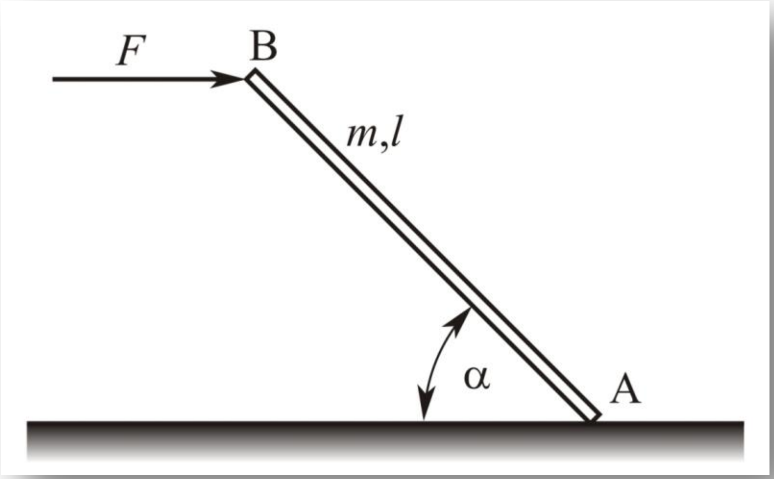
$$\frac{d\vec{L}_C}{dt} = \sum_{i=1}^{N} \vec{M}_C^{\vec{F}_i^s} + \vec{\mathfrak{M}}$$

$$\vec{L}_C = \sum_{i=1}^{N} \vec{\rho}_i \times m_i \vec{v}_{ir} = 0 \quad \Rightarrow \quad \sum_{i=1}^{N} \vec{M}_C^{\vec{F}_i^s} = 0$$

$$\sum_{i=1}^{N} M_{Cx_1}^{\vec{F}_i^s} + \mathfrak{M}_x = 0, \quad \sum_{i=1}^{N} M_{Cy_1}^{\vec{F}_i^s} + \mathfrak{M}_y = 0, \quad \sum_{i=1}^{N} M_{Cz_1}^{\vec{F}_i^s} + \mathfrak{M}_z = 0$$

Primer

- Štap AB, mase m i dužine l, krajem A klizi po glatkoj, horizontalnoj, nepokretnoj podlozi. Na kraj B deluje horizontalna sila konstantnog intenziteta F. Koliki treba da je intenzitet ove sile da bi se štap kretao translatorno, pri čemu gradi ugao α sa horizontalom. Odrediti kretanje štapa u tom slučaju.

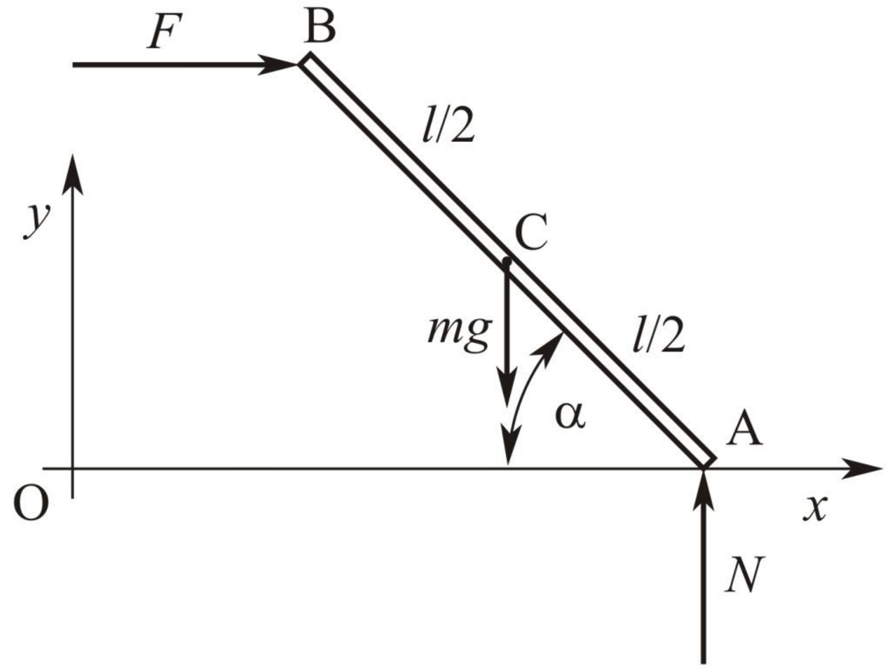


$$M\,\vec{a}_C = \vec{F}_g^s$$

$$m\,\vec{a}_C = \vec{F} + m\vec{g} + \vec{N}$$

$$m\,\ddot{x}_C = F$$

$$m\ddot{y}_C = -mg + N$$



Translatorno kretanje:

$$\alpha = const \rightarrow y_C = l\sin\alpha = const \rightarrow \dot{y}_C = \ddot{y}_C = 0$$

$$N = mg$$

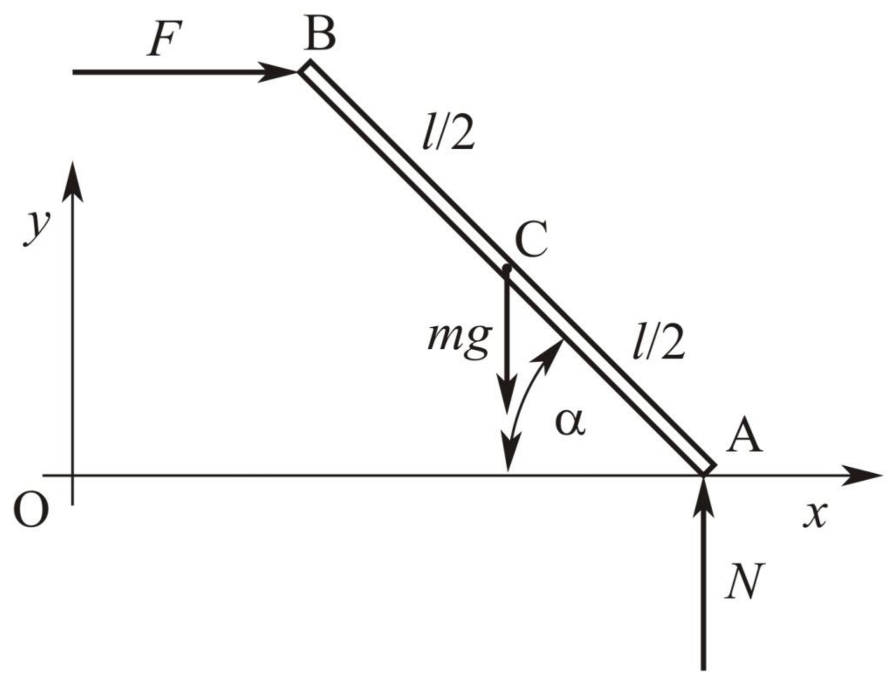
Translatorno kretanje:

$$\sum_{i=1}^{N} \vec{M}_C^{\vec{F}_i^s} = 0$$

$$N\frac{l}{2}\cos\alpha - F\frac{l}{2}\sin\alpha = 0$$

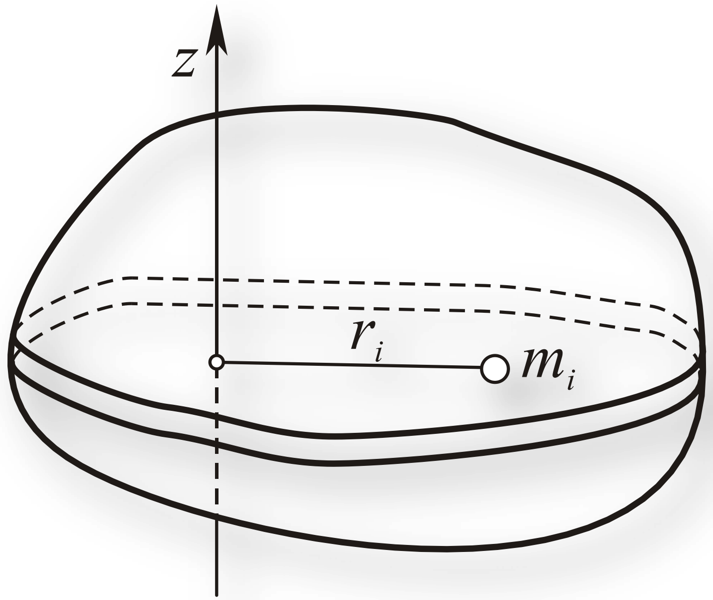
$$F = N\cot\alpha = mg\cot\alpha$$

$$\boxed{m\ddot{x}_C = mg\cot\alpha}$$



$$\ddot{x}_C = g\cot\alpha \rightarrow \dot{x}_C = (g\cot\alpha)t + \dot{x}_{C0} \rightarrow x_C = (g\cot\alpha)\frac{t^2}{2} + \dot{x}_{C0}t + x_{C0}$$

38. Moment inercije tela. Štajnerova teorema



Moment inercije sistema materijalnih tačaka u odnosu na osu z:

$$J_z = \sum_{i=1}^{N} m_i\, r_i^2$$

Moment inercije tela u odnosu na osu z:

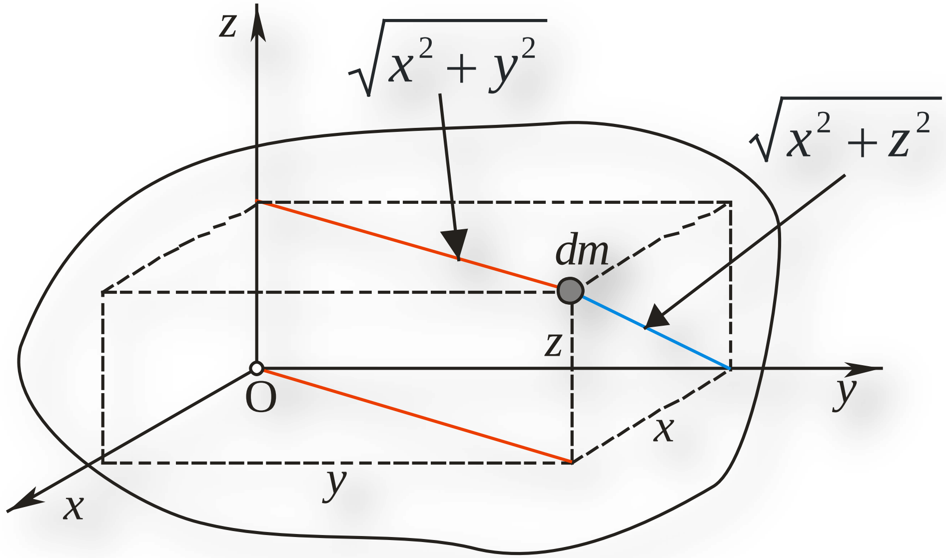
$$\left.\begin{matrix} m_i \to dm \\ r_i \to r \end{matrix}\right\} \to \qquad J_z = \int\limits_{(M)} r^2\, dm$$

$$dm = \rho dV \qquad J_z = \rho \int\limits_{(V)} r^2\, dV$$

Poluprečnik inercije tela u odnosu na osu z:

$$i_z^2 = \frac{J_z}{M} \to J_z = M i_z^2$$

Momenti inercije tela za koordinatne ose

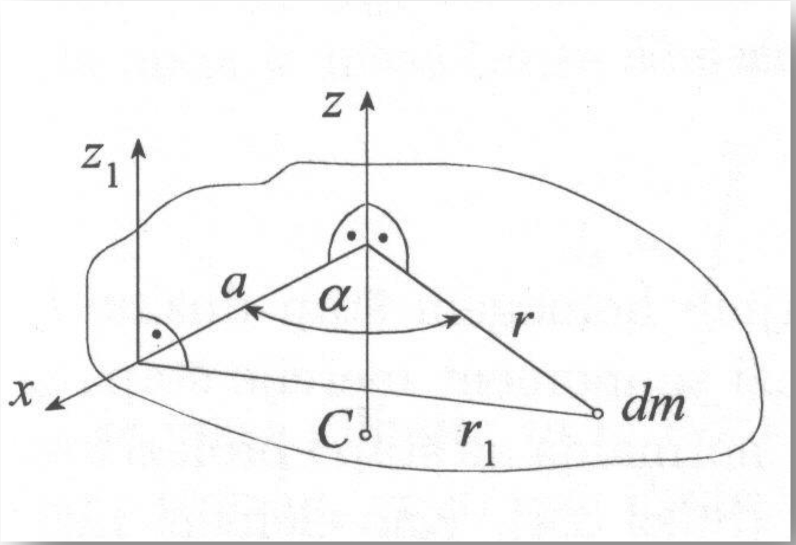


$$J_x = \int_{(M)} (y^2+z^2)\,dm = \rho \int_{(V)} (y^2+z^2)\,dV \qquad J_y = \int_{(M)} (x^2+z^2)\,dm = \rho \int_{(V)} (x^2+z^2)\,dV$$

$$J_z = \int_{(M)} (x^2+y^2)\,dm = \rho \int_{(V)} (x^2+y^2)\,dV$$

Štajnerova teorema – moment inercije za paralelne ose

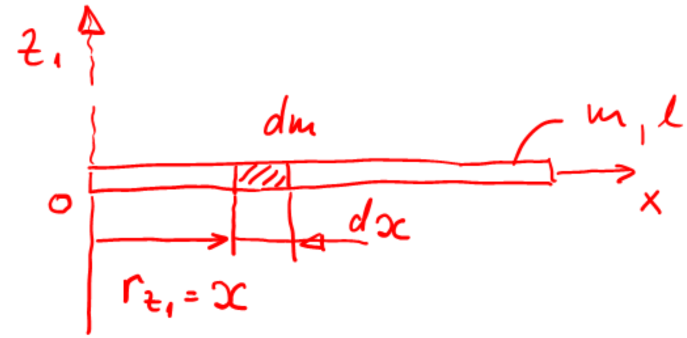
$$J_{z_1} = \int_{(M)} (r_1^2)\, dm$$

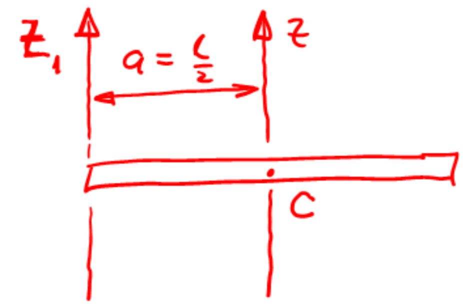


$$J_{z_1} = \int_{(M)} (a^2 + r^2 - 2ar\cos\alpha)\, dm$$

$$J_{z_1} = a^2 \int_{(M)} dm + \int_{(M)} r^2\, dm - 2a \int_{(M)} r\cos\alpha\, dm \qquad \int_{(M)} r\cos\alpha\, dm = 0$$

$$\boxed{J_{z_1} = J_z + Ma^2}$$





J_{z_1}

$$J_{z_1} = \int_m r_{z_1}^2\,dm = \rho\int_0^l x^2\,\rho\,dx = \rho\cdot\frac{x^3}{3}\Big|_0^l$$

$$\rho = \frac{m}{l}\left[\frac{kg}{m}\right]$$

$$m = \rho\,l$$

$$dm = \rho\,dx$$

$$J_{z_1} = \frac{m}{\cancel{l}}\,\frac{l^{\cancel{3}\,2}}{3}$$

$$\boxed{J_{z_1} = \frac{ml^2}{3}}$$

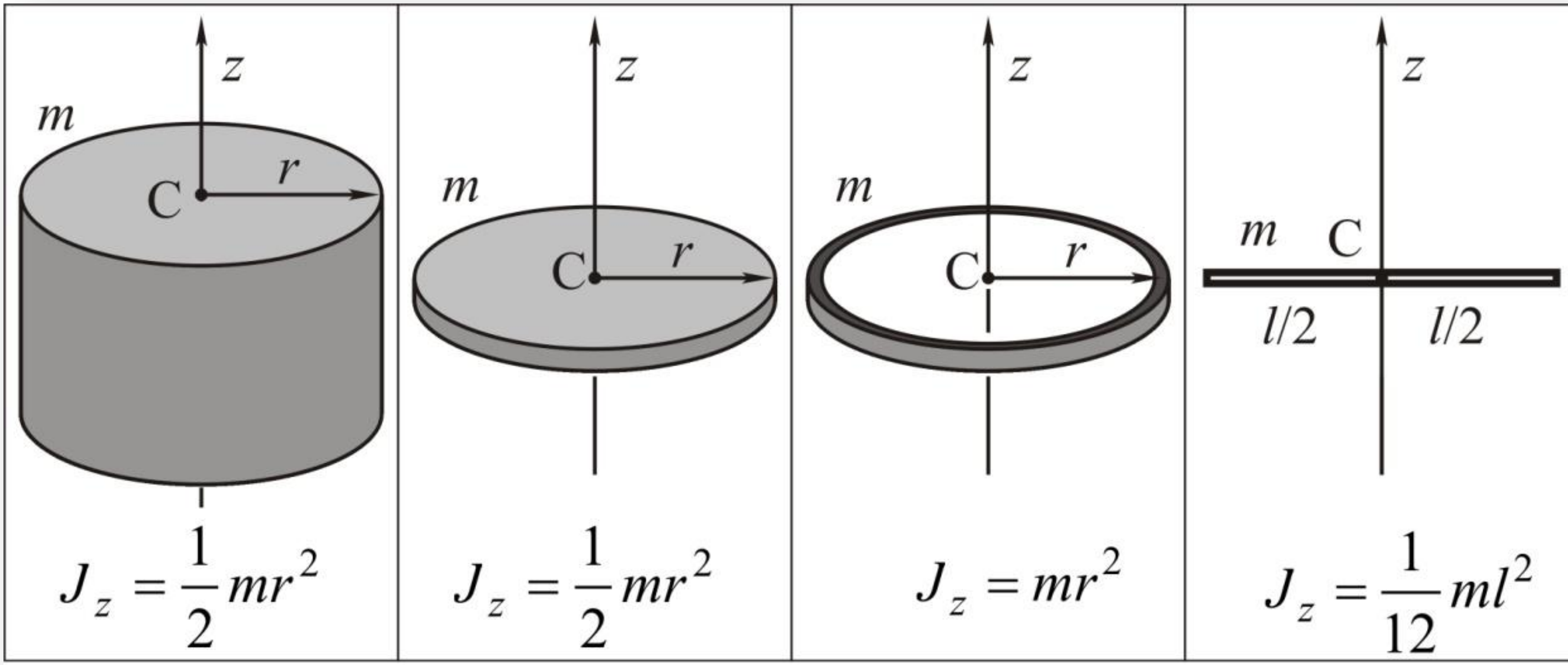
$$J_{z_1} = J_z + m\,a^2$$

$$J_z = J_{z_1} - m\,a^2$$

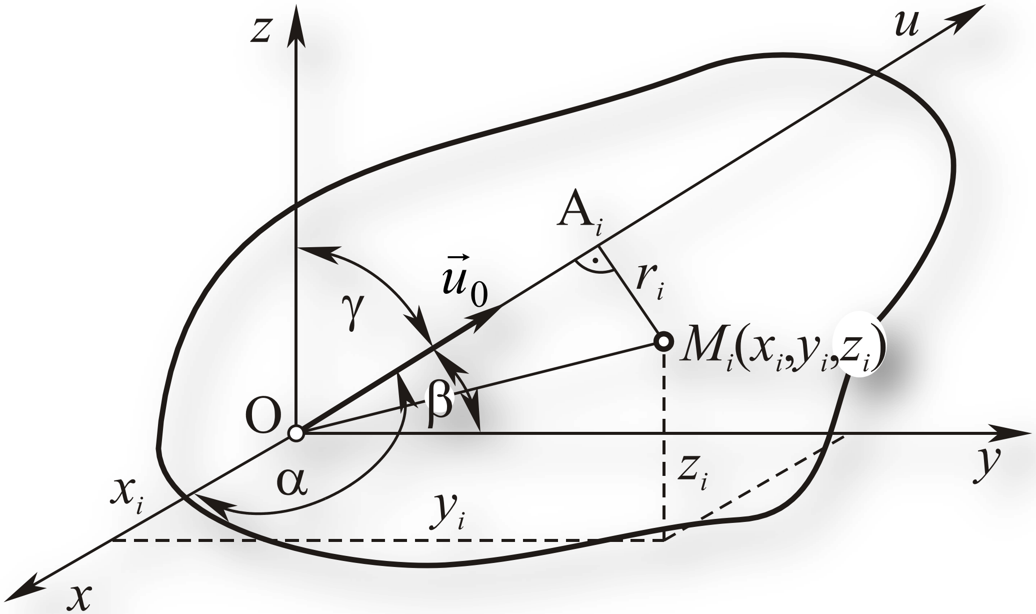
$$J_z = \frac{ml^2}{3} - m\left(\frac{l}{2}\right)^2$$

$$J_z = \frac{ml^2}{12}$$

Moment inercije - primeri



Moment inercije tela u odnosu na proizvoljnu osu



$$J_u = \sum_{i=1}^{N} m_i\, r_i^2 = \sum_{i=1}^{N} m_i\, \overline{A_i M_i}^2$$

$$J_u = J_x \cos^2\alpha + J_y \cos^2\beta + J_z \cos^2\gamma - 2J_{xy}\cos\alpha\cos\beta - 2J_{xz}\cos\alpha\cos\gamma - 2J_{yz}\cos\beta\cos\gamma$$

Centrifugalni moment inercije

MOMENTI INERCIJE:

$$J_x = \int_{(M)} (y^2 + z^2)\, dm$$

$$J_y = \int_{(M)} (x^2 + z^2)\, dm$$

$$J_z = \int_{(M)} (x^2 + y^2)\, dm$$

CENTRIFUGALNI MOMENTI INERCIJE:

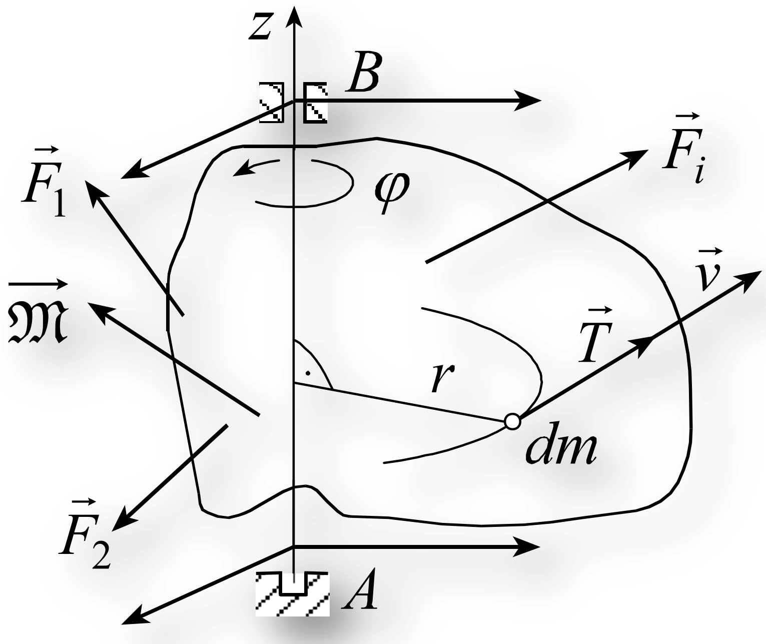
$$\boxed{J_{xy} = J_{yx} = \int_{(M)} (xy)\, dm}$$

$$\boxed{J_{xz} = J_{zx} = \int_{(M)} (xz)\, dm}$$

$$\boxed{J_{yz} = J_{zy} = \int_{(M)} (yz)\, dm}$$

39. Diferencijalna jednačina obrtanja oko nepomične ose. Kinetička energija obrtanja

$$\frac{d\vec{L}_A}{dt} = \sum_{i=1}^{N} \vec{M}_A^{\vec{F}_i} \rightarrow \dot{L}_z = \sum_{i=1}^{N} M_z^{\vec{F}_i} + \mathfrak{M}_z$$



Elementarna količina kretanja

$$dK = dm\, v = dm(r\dot{\varphi})$$

Elementarni moment količine kretanja

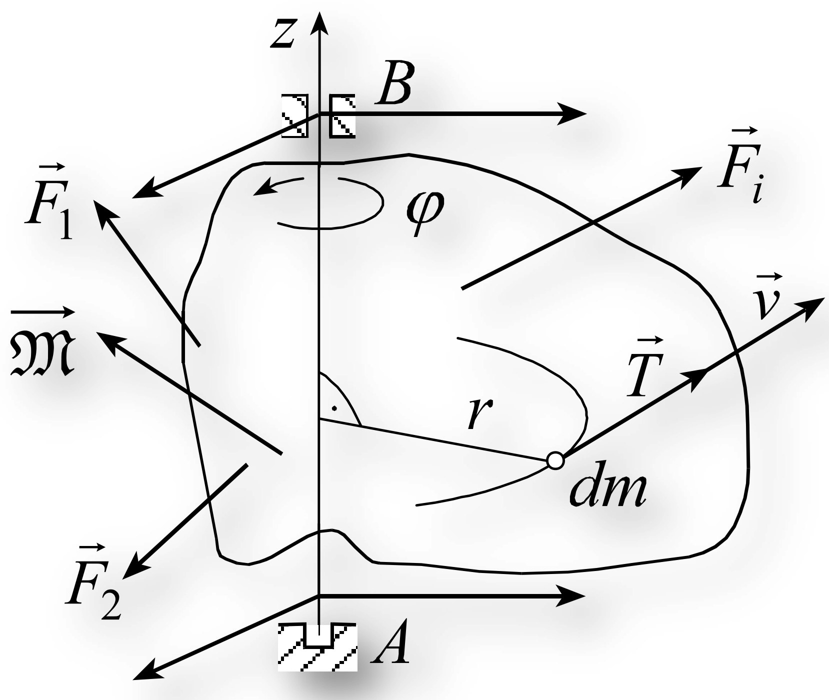
$$dL_z = dK\, r = dm(r\dot{\varphi})r = \dot{\varphi} r^2 dm$$

$$L_z = \int_{(M)} dL_z = \dot{\varphi} \int_{(M)} r^2 dm = \dot{\varphi} J_z$$

Diferencijalna jednačina kretanja

$$\dot{L}_z = \frac{dL_z}{dt} = \frac{d}{dt}(J_z\dot{\varphi}) = J_z\ddot{\varphi} \rightarrow \boxed{J_z\ddot{\varphi} = \sum_{i=1}^{N} M_z^{\vec{F}_i} + \mathfrak{M}_z}$$

Kinetička energija obrtanja



$$dE_k = \frac{1}{2}dmv^2 = \frac{1}{2}dm(r\dot{\varphi})^2 = \frac{1}{2}dmr^2\dot{\varphi}^2$$

$$E_k = \int_{(M)} dE_k = \frac{1}{2}\dot{\varphi}^2 \int_{(M)} r^2 dm = \frac{1}{2}\dot{\varphi}^2 J_z$$

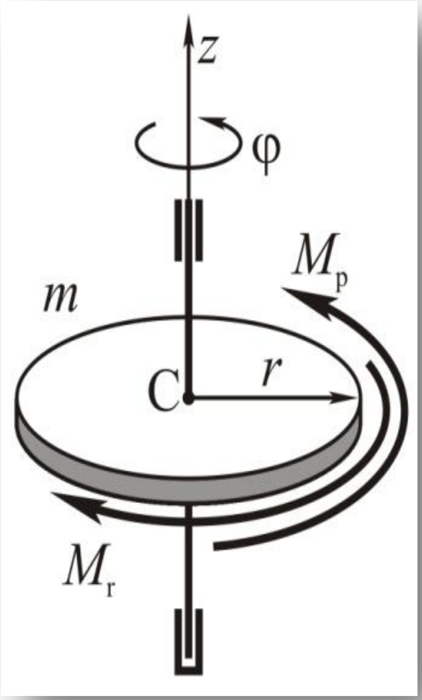
$$\boxed{E_k = \frac{1}{2}J_z\dot{\varphi}^2}$$

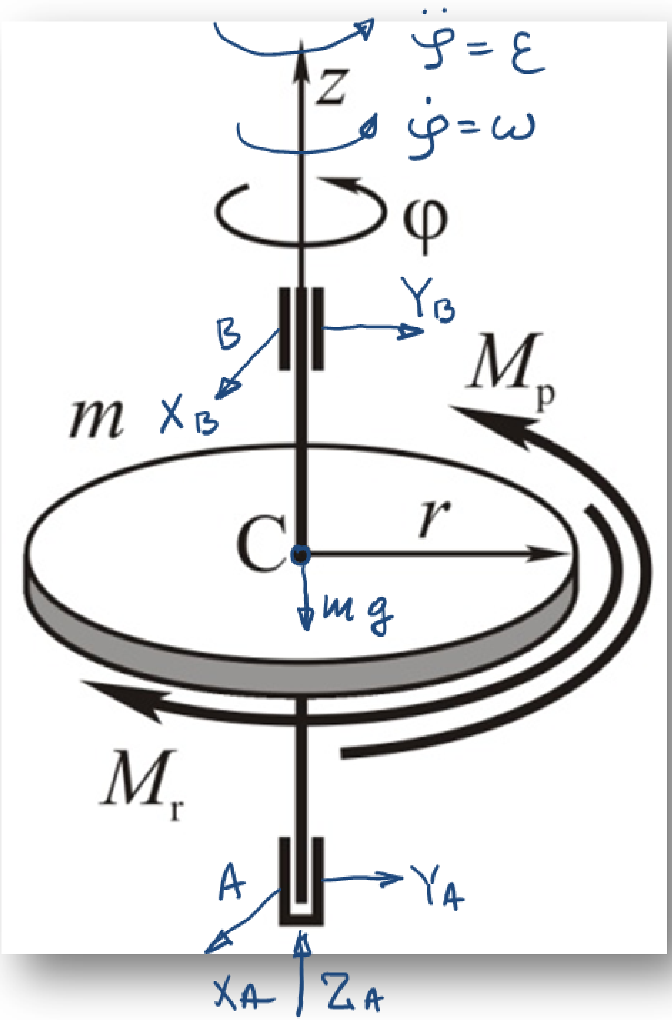
Primer

Disk, mase *m* i poluprečnika *r*, obrće se oko nepokretne ose. Na njega deluje pogonski moment koji je funkcija ugaone brzine

$$M_p = M_0\left(1 - \frac{\dot{\varphi}}{\Omega}\right)$$

i konstantan radni moment M_r. Odrediti zakon promene ugaone brzine diska, koji kreće iz stanja mirovanja.





ДЈК

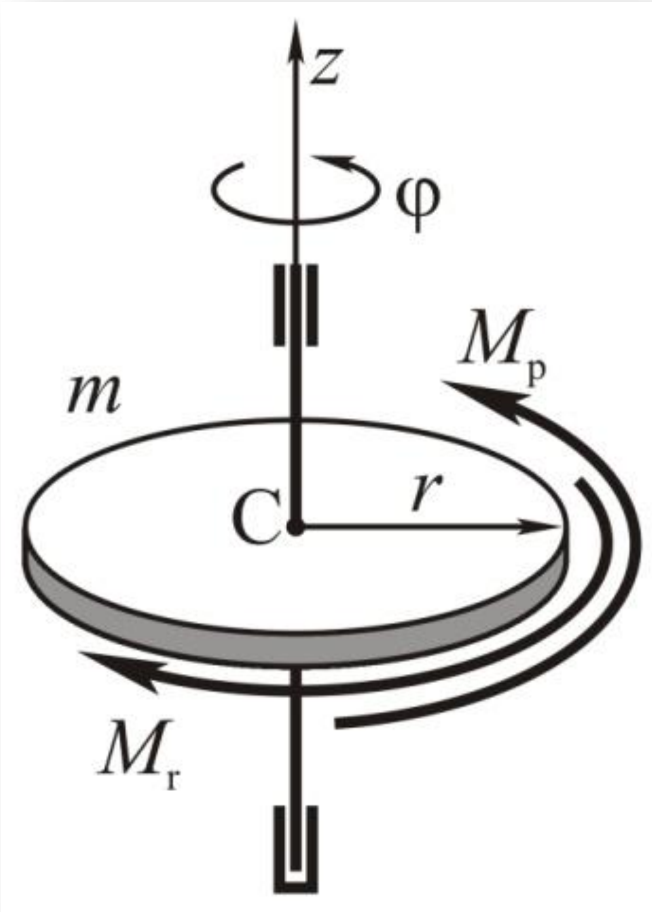
$$J_z \cdot \varepsilon = \sum M_z$$

$$J_z = \frac{mr^2}{2}$$

$$J_z \ddot{\varphi} = M_p - M_r$$

$$J_z \frac{d\dot{\varphi}}{dt} = \left(M_0\left(1 - \frac{\dot{\varphi}}{\Omega}\right) - M_r\right)$$

$$\int_{\dot{\varphi}(0)=0}^{\dot{\varphi}} \frac{J_z \, d\dot{\varphi}}{M_0\left(1 - \frac{\dot{\varphi}}{\Omega}\right) - M_r} = \int_0^t dt$$



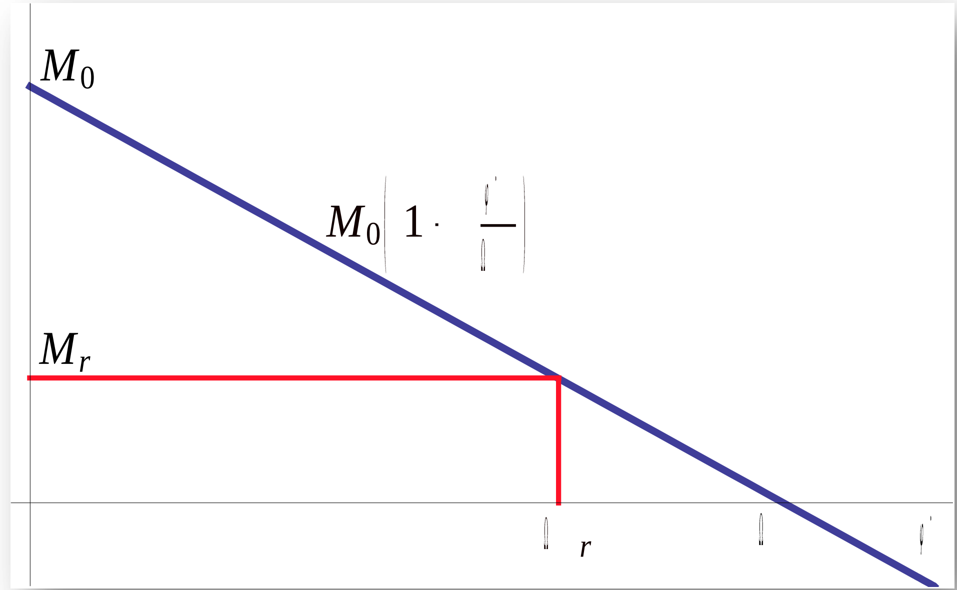
$$J_z\ddot{\varphi} = M_p - M_r; \quad J_z = \frac{mr^2}{2}$$

$$J_z \frac{d\dot{\varphi}}{dt} = M_0\left(1 - \frac{\dot{\varphi}}{\Omega}\right) - M_r$$

$$\int_{\dot{\varphi}(0)=0}^{\dot{\varphi}} \frac{J_z}{M_0\left(1 - \frac{\dot{\varphi}}{\Omega}\right) - M_r} d\dot{\varphi} = \int_0^t dt$$

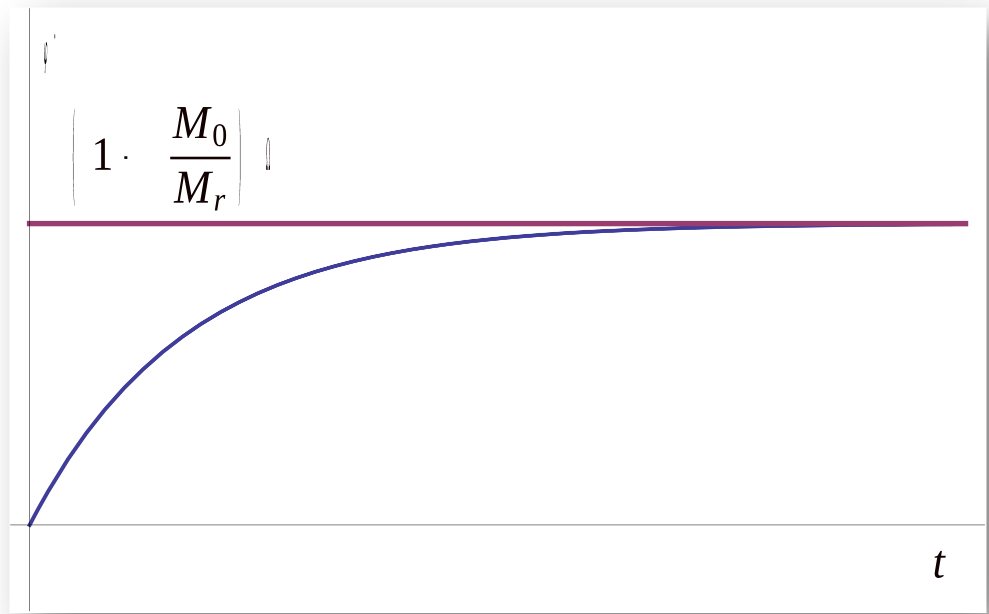
$$-\frac{J_z\Omega}{M_0} \ln\left(\frac{M_0\left(1 - \frac{\dot{\varphi}}{\Omega}\right) - M_r}{M_0 - M_r}\right) = t$$

$$\frac{M_0\left(1-\frac{\dot{\varphi}}{\Omega}\right)-M_r}{M_0-M_r}=e^{-\frac{M_0}{J_z\Omega}t}$$

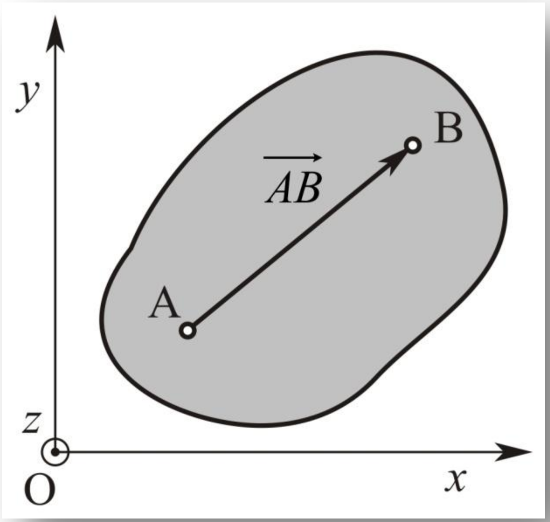


$$\boxed{\dot{\varphi}=\Omega\left(1-\frac{M_r}{M_0}\right)-\Omega\left(1-\frac{M_r}{M_0}\right)e^{-\frac{M_0}{J_z\Omega}t}}$$

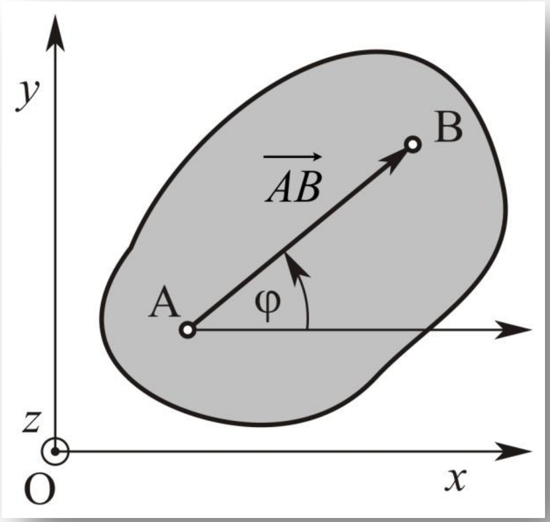
$$\boxed{\lim_{t\to\infty}\dot{\varphi}=\Omega\left(1-\frac{M_r}{M_0}\right)=\Omega_r}$$



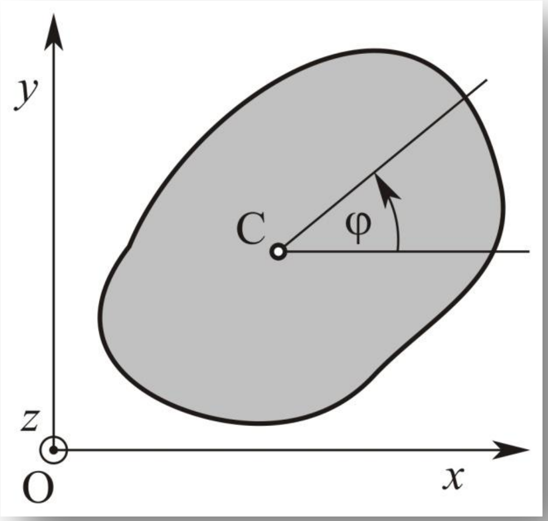
40. Diferencijalne jednačine ravanskog kretanja tela. Kinetička energija



$$x_A, y_A, x_B, y_B$$



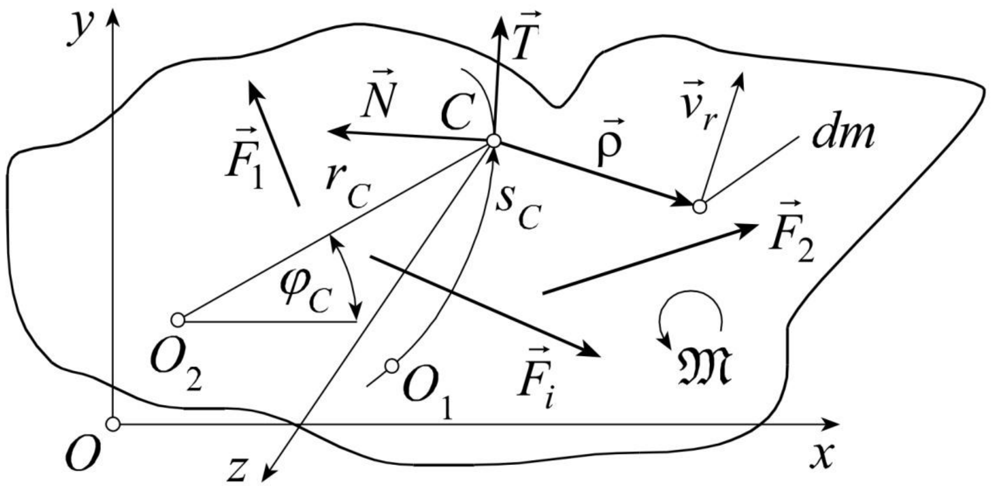
$$x_A, y_A, \varphi$$



$$(x_A - x_B)^2 + (y_A - y_B)^2 = const$$

$$x_C, y_C, \varphi$$

Diferencijalne jednačine ravanskog kretanja tela



$$M\,\vec{a}_C = \sum \vec{F}_i^s$$

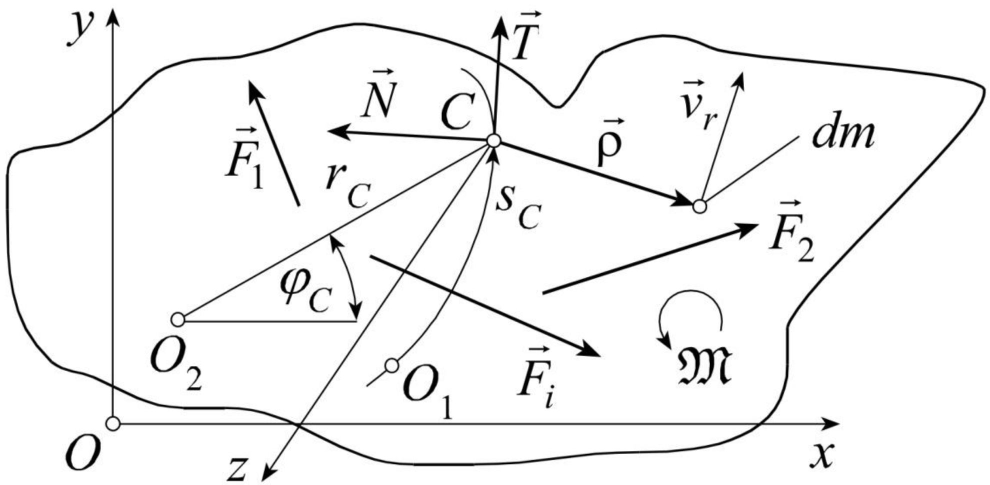
$$M\,\ddot{x}_C = \sum F_{ix}^s$$
$$M\,\ddot{y}_C = \sum F_{iy}^s$$

$$\frac{d\vec{L}_C}{dt} = \sum \vec{M}_C^{\vec{F}_i^s} + \vec{\mathfrak{M}} \rightarrow \frac{dL_{Cz}}{dt} = \sum M_{Cz}^{\vec{F}_i^s} + \mathfrak{M}_{Cz}$$

$$L_{Cz} = L_z = L_C = J_{Cz}\dot{\varphi} = J_C\dot{\varphi}$$
$$\frac{dL_{Cz}}{dt} = J_C\ddot{\varphi}$$

$$J_C\ddot{\varphi} = \sum M_C$$

Diferencijalne jednačine ravanskog kretanja tela



$$\frac{d\vec{L}_C}{dt} = \sum \vec{M}_C^{\vec{F}_i^s} + \vec{\mathfrak{M}}$$

$$J_C \ddot{\varphi} = \sum M_C$$

$$M\,\vec{a}_C = \sum \vec{F}_i^s$$

$$M\,\vec{a}_C = \sum \vec{F}_i^s$$

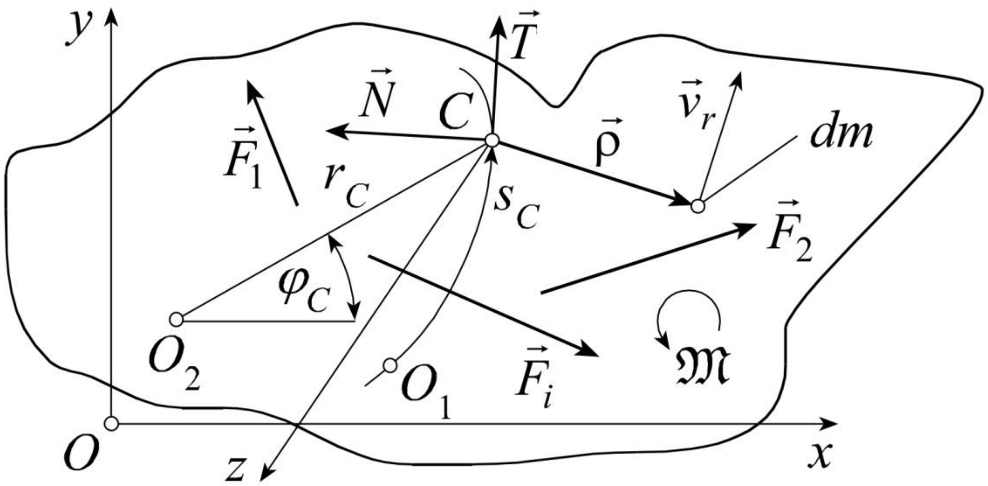
$$M\,\vec{a}_C = \sum \vec{F}_i^s$$

$$M\,\ddot{x}_C = \sum F_{ix}^s$$
$$M\,\ddot{y}_C = \sum F_{iy}^s$$

$$M\,\ddot{s}_C = \sum F_{iT}^s$$
$$M\,\frac{\dot{s}_C^2}{R_{KC}} = \sum F_{iT}^s$$

$$M\left(\ddot{r}_C - r_C\dot{\varphi}^2\right) = \sum F_{ir}^s$$
$$M\left(r_C\ddot{\varphi} + 2\dot{r}_C\dot{\varphi}\right) = \sum F_{ic}^s$$

Kinetička energija



$$E_K = \frac{1}{2} M v_C^2 + \frac{1}{2} \sum_{i=1}^{N} m_i (v_{ir}^C)^2$$

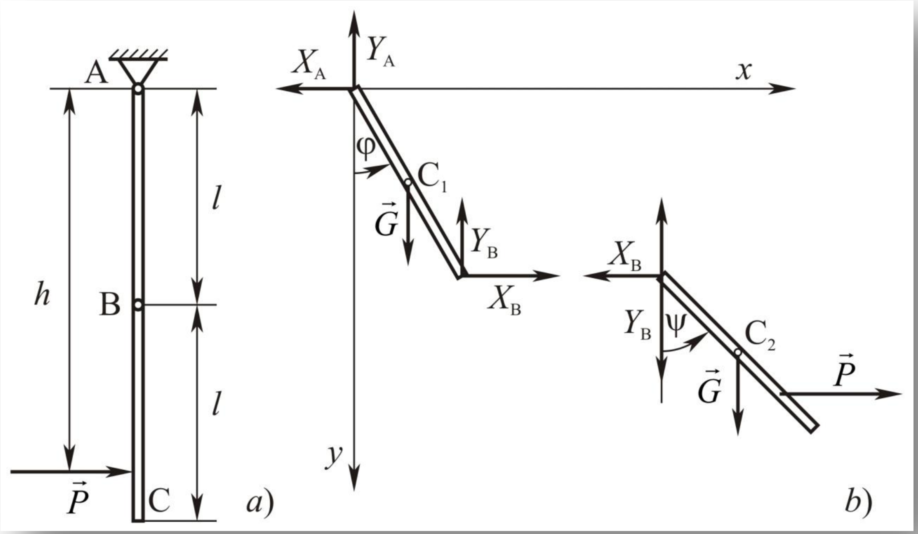
$$E_K = \frac{1}{2} M v_C^2 + E_{Kr}^C$$

$$E_{Kr}^C = \frac{1}{2} \sum_{i=1}^{N} m_i (\rho_i \dot{\varphi})^2 = \frac{1}{2} \left(\sum_{i=1}^{N} m_i (\rho_i)^2 \right) \dot{\varphi}^2 = \frac{1}{2} \left(\int_M \rho^2 dm \right) \dot{\varphi}^2 = \frac{1}{2} J_C \dot{\varphi}^2$$

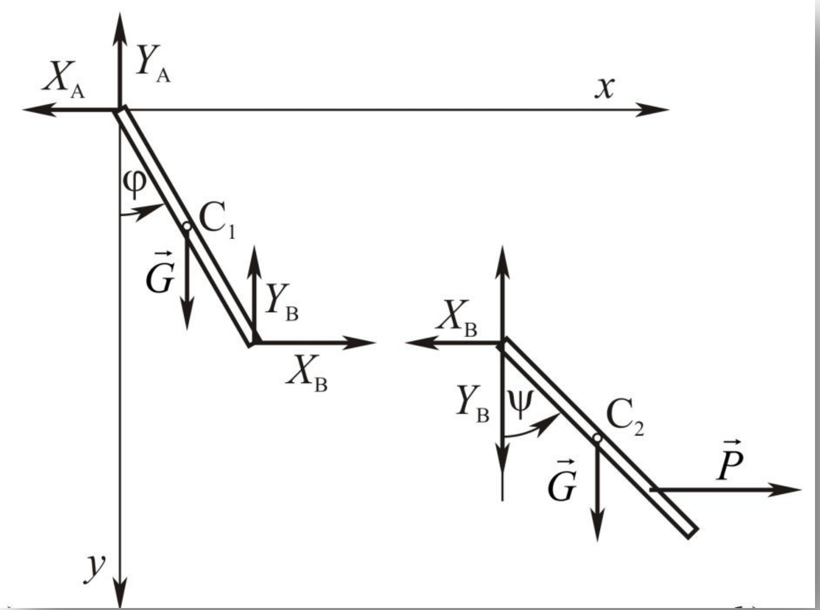
$$\boxed{E_K = \frac{1}{2} M v_C^2 + \frac{1}{2} J_C \dot{\varphi}^2}$$

41. Ravansko kretanje sistema krutih tela

Diferencijalne jednačine kretanja:



$$
\begin{aligned}
m\ddot{x}_{C_1} &= -X_A + X_B,\\
m\ddot{y}_{C_1} &= -Y_A - Y_B + G,\\
J_{C_1}\ddot{\varphi} &= X_A\frac{L}{2}\cos\varphi - Y_A\frac{L}{2}\sin\varphi + Y_B\frac{L}{2}\sin\varphi + X_B\frac{L}{2}\cos\varphi,\\
m\ddot{x}_{C_2} &= -X_B + P,\\
m\ddot{y}_{C_2} &= Y_B + G,\\
J_{C_2}\ddot{\psi} &= X_B\frac{L}{2}\cos\psi + Y_B\frac{L}{2}\sin\psi + P(h - \frac{3L}{2})\cos\psi.
\end{aligned}
$$



Jednačine veza:

$$
\begin{aligned}
x_{C_1} &= \frac{L}{2}\sin\varphi, \quad x_{C_2} = L\sin\varphi + \frac{L}{2}\sin\psi, \\
y_{C_1} &= \frac{L}{2}\cos\varphi, \quad y_{C_2} = L\cos\varphi + \frac{L}{2}\cos\psi.
\end{aligned}
$$

$$
\begin{aligned}
\ddot{x}_{C_1} &= \frac{L}{2}\ddot{\varphi}\cos\varphi - \frac{L}{2}\dot{\varphi}^2\sin\varphi, \quad \ddot{y}_{C_1} = -\frac{L}{2}\ddot{\varphi}\sin\varphi - \frac{L}{2}\dot{\varphi}^2\cos\varphi, \\
\ddot{x}_{C_2} &= L\ddot{\varphi}\cos\varphi - L\dot{\varphi}^2\sin\varphi + \frac{L}{2}\ddot{\psi}\cos\psi - \frac{L}{2}\dot{\psi}^2\sin\psi, \\
\ddot{y}_{C_2} &= -L\ddot{\varphi}\sin\varphi - L\dot{\varphi}^2\cos\varphi - \frac{L}{2}\ddot{\psi}\sin\psi - \frac{L}{2}\dot{\psi}^2\cos\psi.
\end{aligned}
$$

Diferencijalne jednačine kretanja:

$$
\begin{aligned}
m\ddot{x}_{C_1} &= -X_A + X_B, \\
m\ddot{y}_{C_1} &= -Y_A - Y_B + G, \\
J_{C_1}\ddot{\varphi} &= X_A\frac{L}{2}\cos\varphi - Y_A\frac{L}{2}\sin\varphi + Y_B\frac{L}{2}\sin\varphi + X_B\frac{L}{2}\cos\varphi, \\
m\ddot{x}_{C_2} &= -X_B + P, \\
m\ddot{y}_{C_2} &= Y_B + G, \\
J_{C_2}\ddot{\psi} &= X_B\frac{L}{2}\cos\psi + Y_B\frac{L}{2}\sin\psi + P\left(h - \frac{3L}{2}\right)\cos\psi.
\end{aligned}
$$

Jednačine veza:

$$
\begin{aligned}
\ddot{x}_{C_1} &= \frac{L}{2}\ddot{\varphi}\cos\varphi - \frac{L}{2}\dot{\varphi}^2\sin\varphi, \quad \ddot{y}_{C_1} = -\frac{L}{2}\ddot{\varphi}\sin\varphi - \frac{L}{2}\dot{\varphi}^2\cos\varphi, \\
\ddot{x}_{C_2} &= L\ddot{\varphi}\cos\varphi - L\dot{\varphi}^2\sin\varphi + \frac{L}{2}\ddot{\psi}\cos\psi - \frac{L}{2}\dot{\psi}^2\sin\psi, \\
\ddot{y}_{C_2} &= -L\ddot{\varphi}\sin\varphi - L\dot{\varphi}^2\cos\varphi - \frac{L}{2}\ddot{\psi}\sin\psi - \frac{L}{2}\dot{\psi}^2\cos\psi.
\end{aligned}
$$

$$
\Rightarrow
\begin{bmatrix} \ddot{x}_{C1} \\ \ddot{y}_{C1} \\ \ddot{\varphi} \\ \ddot{x}_{C2} \\ \ddot{y}_{C2} \\ \ddot{\psi} \\ X_A \\ Y_A \\ X_B \\ Y_B \end{bmatrix}
\xrightarrow[\text{PU}]{\text{Int.}}
\begin{bmatrix} x_{C1} \\ y_{C1} \\ \varphi \\ x_{C2} \\ y_{C2} \\ \psi \\ X_A \\ Y_A \\ X_B \\ Y_B \end{bmatrix}
$$

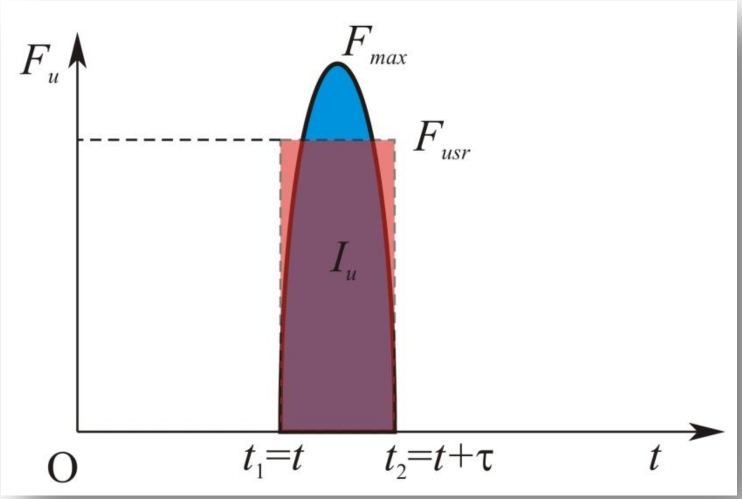
42. Osnove teorije udara

Osnovi teorije udara materijalne tačke

- Impuls udarne sile

$$\boxed{\vec{I}_u = \int_{t_1}^{t_2 = t_1 + \tau} \vec{F}_u \, dt}$$

$$\tau = 0.0001 - 0.01 [s]$$



- Zakon o promeni količine kretanja (jednačina udara)

$$\boxed{m\vec{v}_2 - m\vec{v}_1 = \vec{I}_u}$$

[redacted]

[redacted]

[redacted]

[redacted]

Osnovi teorije udara materijalne tačke

- Pri dejstvu udara impulsi neudarnih sila i spregova se zanemaruju.
- Za vreme udara pomeranje materijalne tačke ili krutog tela je zanemarljivo.

$$\boxed{\Delta\vec{r} = \vec{r}_2 - \vec{r}_1 = \frac{\vec{v}_2 + \vec{v}_1}{2}\tau \to \vec{r}_2 \approx \vec{r}_1}$$

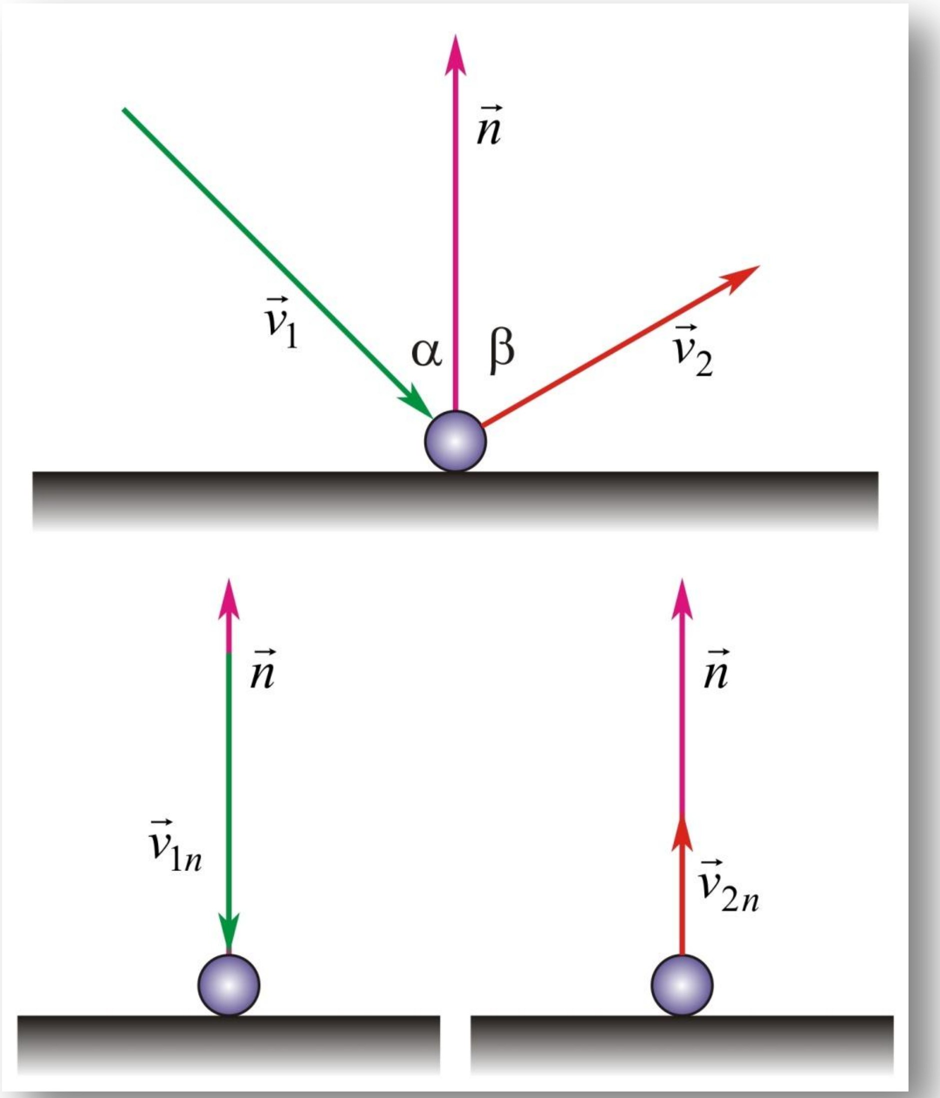
- Za vreme udara dolazi do konačne promene brzine materijalne tačke i ugaone brzine krutog tela.
- Za vreme udara dolazi do konačne promene kinetičke energije materijalne tačke ili krutog tela.

$$\boxed{\Delta E_k = E_{k2} - E_{k1} = A_{12} = \frac{\vec{v}_1 + \vec{v}_2}{2} \cdot \vec{I}_u}$$

Osnovi teorije udara materijalne tačke

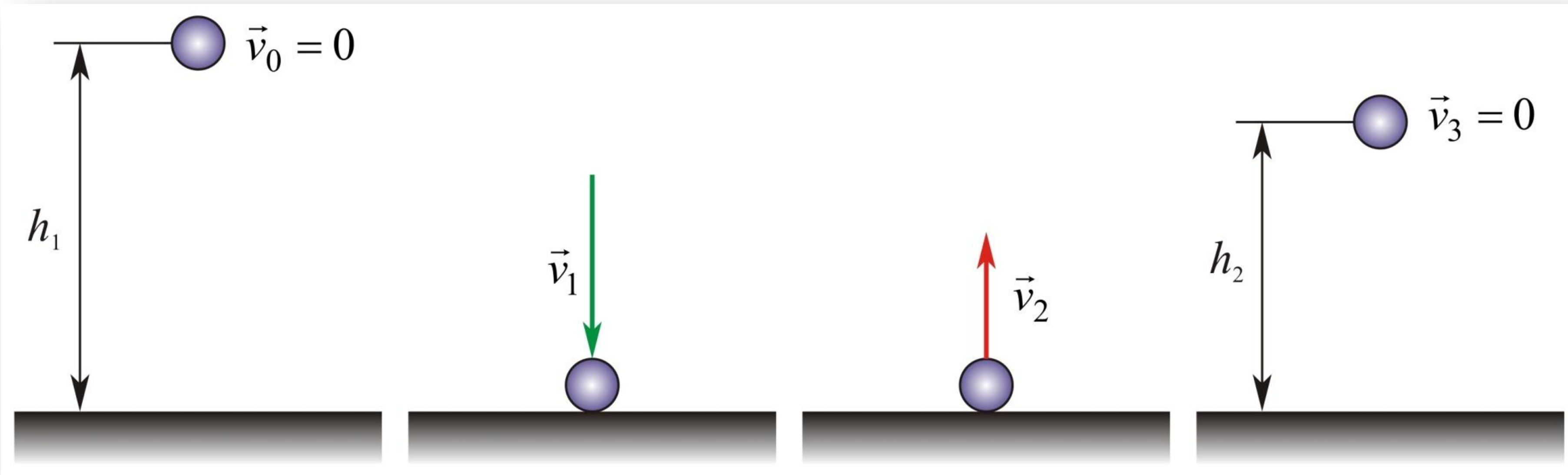
- Zbog kratkog vremenskog intervala u kome deluju udarne sile, udarni impulsi imaju isti pravac sa udarnim silama i spregovima, koji mogu biti aktivni ili reakcije veza.
- Karakteristične veličine pri udaru zavise i od čitavog niza materijalnih svojstava tela, na primer od vrste materijala, oblika tela, modula elastičnosti tela itd.

Koeficijent uspostavljanja - udara



$$k = \frac{|v_{2n}|}{|v_{1n}|} = \frac{|\vec{v}_{2n}|}{|\vec{v}_{1n}|} = \frac{|\vec{v}_2 \cdot \vec{n}|}{|\vec{v}_1 \cdot \vec{n}|}$$

$$0 \leq k \leq 1$$

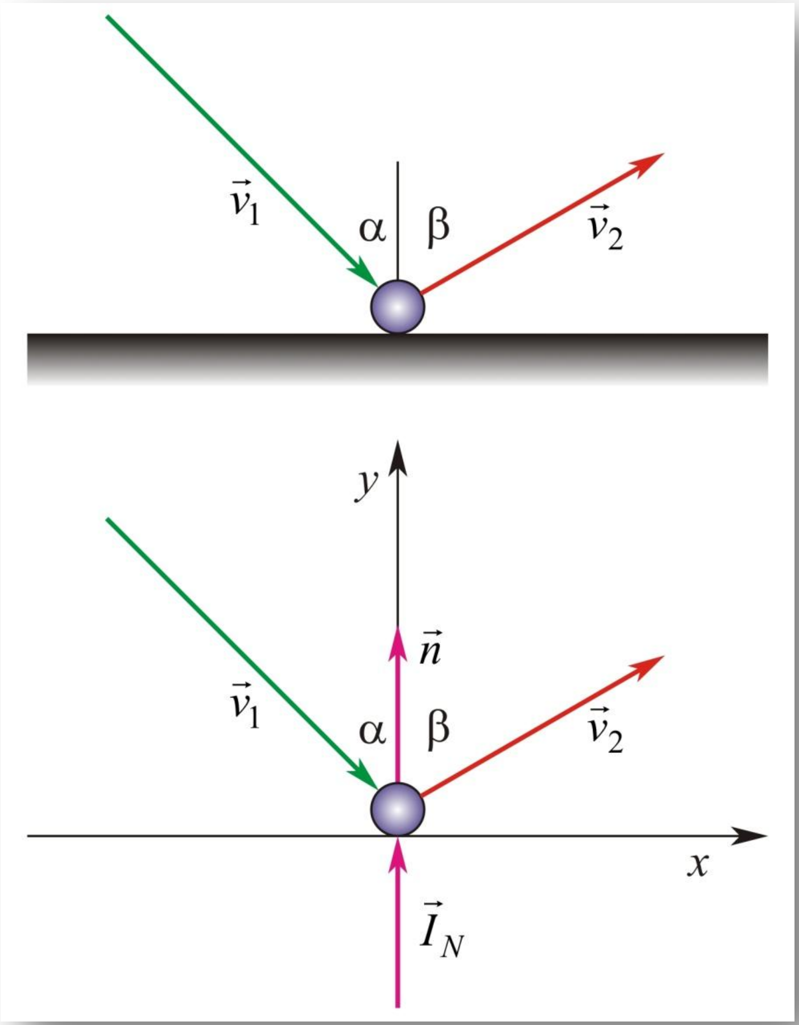


$$E_{k1} + \Pi_1 = E_{k0} + \Pi_0$$
$$\frac{1}{2} m v_1^2 = m g h_1$$
$$v_1 = \sqrt{2 g h_1}$$

$$k = \frac{v_2}{v_1}$$
$$k = \sqrt{\frac{h_2}{h_1}}$$

$$E_{k3} + \Pi_3 = E_{k2} + \Pi_2$$
$$m g h_2 = \frac{1}{2} m v_2^2$$
$$v_2 = \sqrt{2 g h_2}$$

Udar materijalne tačke o glatku pregradu



$$m\vec{v}_2 - m\vec{v}_1 = \vec{I}_u$$

$$(1) \quad mv_2 \sin\beta - mv_1 \sin\alpha = 0$$

$$(2) \quad mv_2 \cos\beta - m(-v_1 \cos\alpha) = I_N$$

$$(3) \quad k = \frac{|v_{2y}|}{|v_{1y}|} = \frac{v_2 \cos\beta}{v_1 \cos\alpha}$$

$$\boxed{I_N = m(1+k)v_1 \cos\alpha}$$

$$\boxed{\tan\beta = \frac{1}{k}\tan\alpha}$$

$$\boxed{v_2 = v_1\sqrt{1-(1-k^2)\cos^2\alpha}}$$

Udar sistema materijalnih tačaka

$$m_1\vec{v}_{1(2)} - m_1\vec{v}_{1(1)} = \vec{I}_1^{(s)} + \vec{I}_1^{(u)}$$

$$m_2\vec{v}_{2(2)} - m_2\vec{v}_{2(1)} = \vec{I}_2^{(s)} + \vec{I}_2^{(u)}$$

$$\ldots$$

$$m_N\vec{v}_{N(2)} - m_N\vec{v}_{N(1)} = \vec{I}_N^{(s)} + \vec{I}_N^{(u)}$$

$$\sum_{i=1}^{N} m_i\vec{v}_{i(2)} - \sum_{i=1}^{N} m_i\vec{v}_{i(1)} = \sum_{i=1}^{N} \vec{I}_i^{(s)} + \sum_{i=1}^{N} \vec{I}_i^{(u)}$$

$$\vec{K}_{(2)} - \vec{K}_{(1)} = \sum_{i=1}^{N} \vec{I}_i^{(s)}$$

$$M\,\vec{v}_{C(2)} - M\,\vec{v}_{C(1)} = \sum_{i=1}^{N} \vec{I}_i^{(s)}$$

$$M\left(\vec{v}_{C(2)} - \vec{v}_{C(1)}\right) = \sum_{i=1}^{N} \vec{I}_i^{(s)}$$

Udar sistema materijalnih tačaka

$$\vec{r}_1 \times m_1\vec{v}_{1(2)} - \vec{r}_1 \times m_1\vec{v}_{1(1)} = \vec{r}_1 \times \vec{I}_1^{(s)} + \vec{r}_1 \times \vec{I}_1^{(u)}$$

$$\vec{r}_2 \times m_2\vec{v}_{2(2)} - \vec{r}_2 \times m_2\vec{v}_{2(1)} = \vec{r}_2 \times \vec{I}_2^{(s)} + \vec{r}_2 \times \vec{I}_2^{(u)}$$

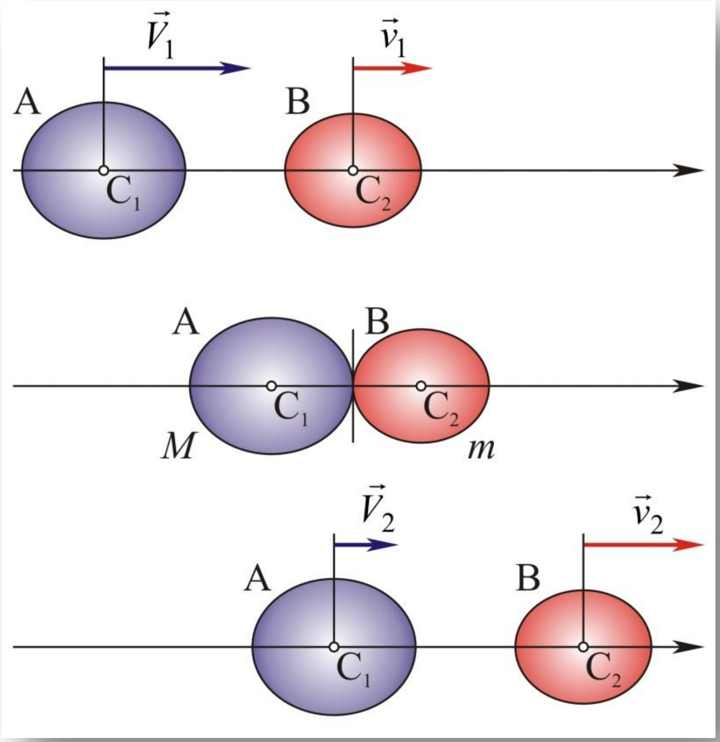
$$\ldots$$

$$\vec{r}_N \times m_N\vec{v}_{N(2)} - \vec{r}_N \times m_N\vec{v}_{N(1)} = \vec{r}_N \times \vec{I}_N^{(s)} + \vec{r}_N \times \vec{I}_N^{(u)}$$

$$\sum_{i=1}^{N} \vec{r}_i \times m_i\vec{v}_{i(2)} - \sum_{i=1}^{N} \vec{r}_i \times m_i\vec{v}_{i(1)} = \sum_{i=1}^{N} \vec{r}_i \times \vec{I}_i^{(s)} + \sum_{i=1}^{N} \vec{r}_i \times \vec{I}_i^{(u)}$$

$$\vec{L}_{O(2)} - \vec{L}_{O(1)} = \sum_{i=1}^{N} \vec{M}_O^{\vec{I}_i^{(s)}}$$

Centralni sudar dva tela



$$\vec{K}_2 - \vec{K}_1 = \vec{I}^{(s)}$$

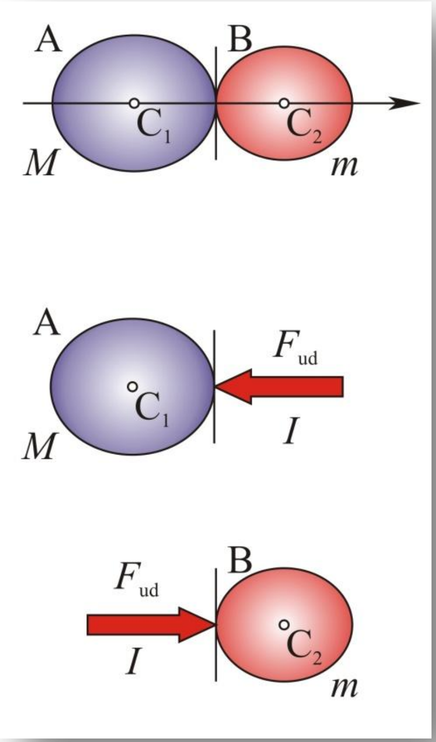
$$\vec{K}_2 - \vec{K}_1 = 0$$

$$\vec{K}_2 = \vec{K}_1$$

$$M\vec{V}_2 + m\vec{v}_2 = M\vec{V}_1 + m\vec{v}_1$$

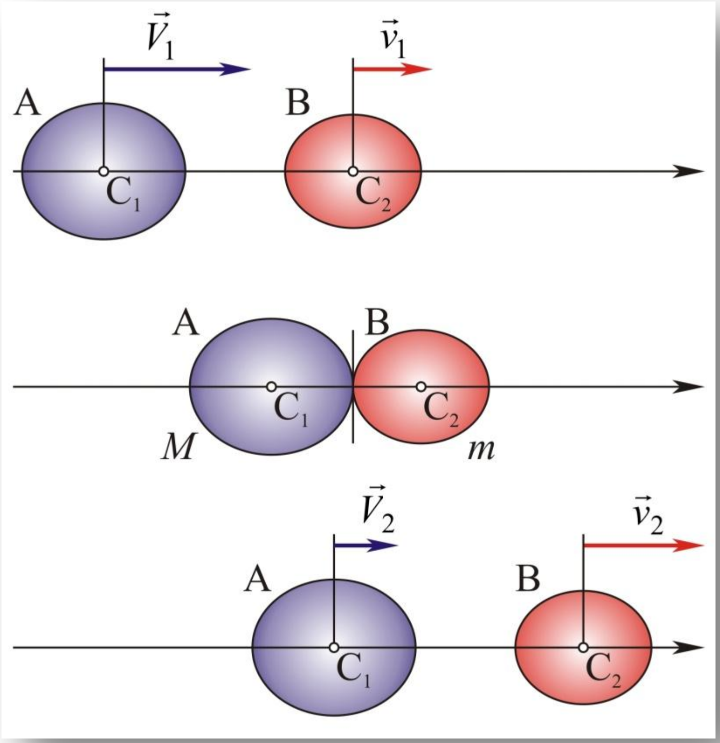
$$\boxed{MV_2 + mv_2 = MV_1 + mv_1}$$

$$\boxed{k = \frac{v_2 - V_2}{V_1 - v_1}}$$



$$m\vec{v}_2 - m\vec{v}_1 = \vec{I}$$

$$\boxed{mv_2 - mv_1 = I}$$



$$\boxed{MV_2 + mv_2 = MV_1 + mv_1}$$

$$\boxed{k = \frac{v_2 - V_2}{V_1 - v_1}}$$

$$\boxed{mv_2 - mv_1 = I}$$

$$V_2 = \frac{M - km}{M + m} V_1 + \frac{m(1+k)}{M + m} v_1$$

$$v_2 = \frac{M(1+k)}{M + m} V_1 + \frac{m - kM}{M + m} v_1$$

$$I = (1+k) \frac{Mm}{M + m} (V_1 - v_1)$$

$$V_2 = \frac{M - km}{M + m} V_1 + \frac{m(1+k)}{M + m} v_1$$

$$v_2 = \frac{M(1+k)}{M + m} V_1 + \frac{m - kM}{M + m} v_1$$

$$I = (1+k)\frac{Mm}{M + m}(V_1 - v_1)$$

$k = 1$

$$V_2 = \frac{M - m}{M + m} V_1 + \frac{2m}{M + m} v_1$$

$$v_2 = \frac{2M}{M + m} V_1 + \frac{m - M}{M + m} v_1$$

$$I = 2\frac{Mm}{M + m}(V_1 - v_1)$$

$k = 0$

$$V_2 = v_2 = \frac{M}{M + m} V_1 + \frac{m}{M + m} v_1$$

$$I = \frac{Mm}{M + m}(V_1 - v_1)$$

$k = 1, M = m$

$$V_2 = v_1$$

$$v_2 = V_1$$

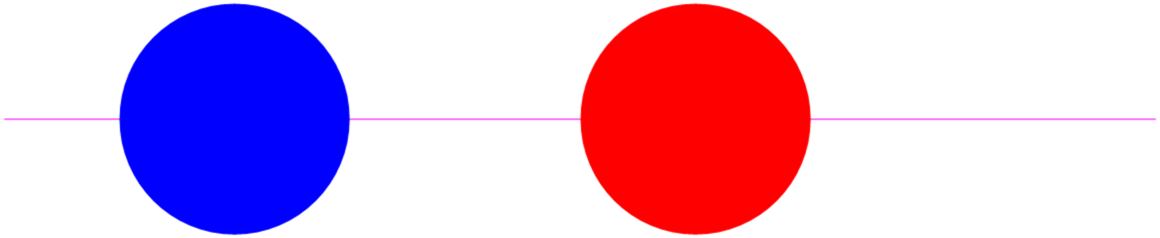
$$I = m(V_1 - v_1)$$

$k = 0, M = m$

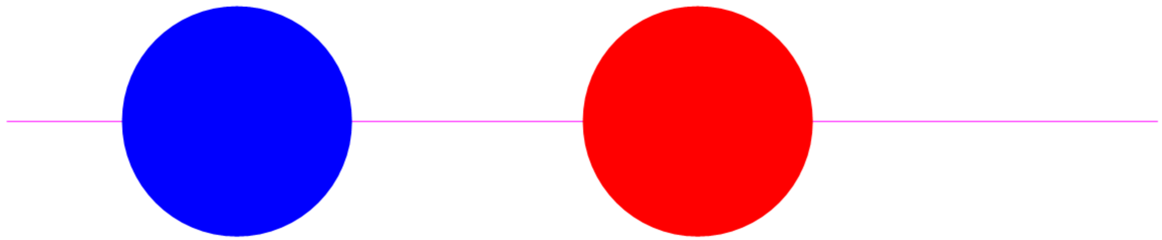
$$V_2 = v_2 = \frac{1}{2} V_1 + \frac{1}{2} v_1$$

$$I = \frac{m}{2}(V_1 - v_1)$$

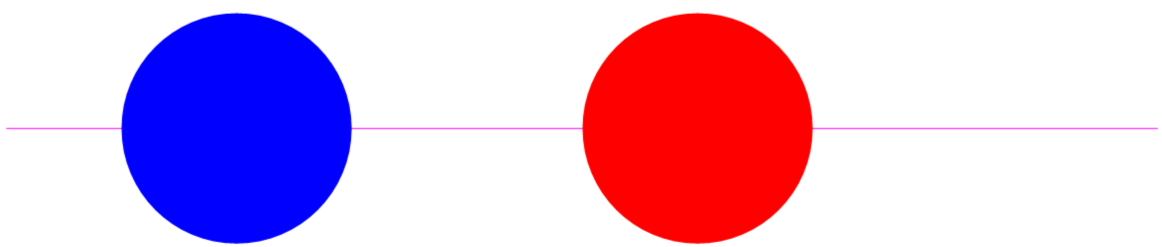
$M=m,\ v_1=0,\ k=1$



$M=m,\ v_1=0,\ k=1/2$



$M=m,\ v_1=0,\ k=0$



Centralni sudar dva tela – gubitak kinetičke energije

$$\Delta E_K = E_{K2} - E_{K1}$$

$$\Delta E_K = \left(\frac{MV_2^2}{2} + \frac{mv_2^2}{2} \right) - \left(\frac{MV_1^2}{2} + \frac{mv_1^2}{2} \right)$$

$$\Delta E_K = -\frac{1-k^2}{2} \frac{Mm}{M+m} \left(V_1 - v_1 \right)^2$$

$$\eta = \left| \frac{\Delta E_K}{E_{K1}} \right| = \left| \frac{E_{K2} - E_{K1}}{E_{K1}} \right| = \left| \frac{E_{K2}}{E_{K1}} - 1 \right| = 1 - \frac{E_{K2}}{E_{K1}}$$

Šta smo naučili?

37. **Translatorno kretanje tela.**
38. **Moment inercije tela. Štajnerova teorema.**
39. **Diferencijalna jednačina obrtanja oko nepomične ose. Kinetička energija obrtanja.**
40. **Diferencijalne jednačine ravanskog kretanja tela. Kinetička energija.**
41. **Ravansko kretanje sistema krutih tela.**
42. **Osnove teorije udara.**

Dinamika
Dinamika krutog tela,…

Kinematika i dinamika

Miodrag Zuković

Novi Sad, 2021.