

Kinematika

Kinematika tačke – 1. deo

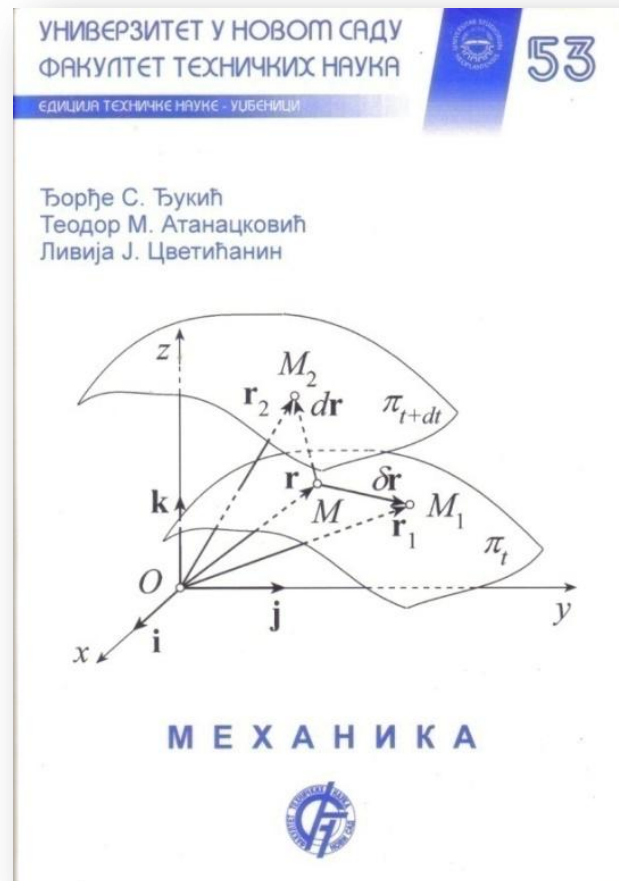
Kinematika i dinamika

Miodrag Zuković

Novi Sad, 2021.

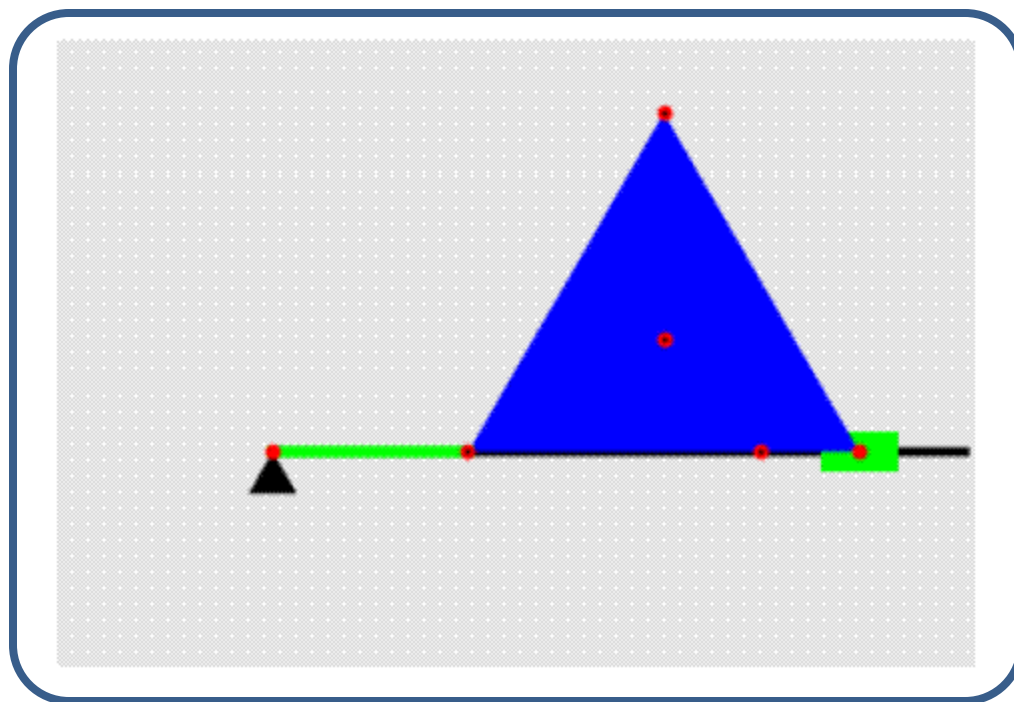
Literatura

- Đorđe S. Đukić, Teodor M. Atanacković, Livija J. Cvetićanin:
Mehanika, Fakultet tehničkih nauka u Novom Sadu, Novi Sad, 2003.



Kinematika

Deo mehanike koji se bavi izučavanjem kretanja tela ne uzimajući u obzir materijalnost i međusobno dejstvo između tela



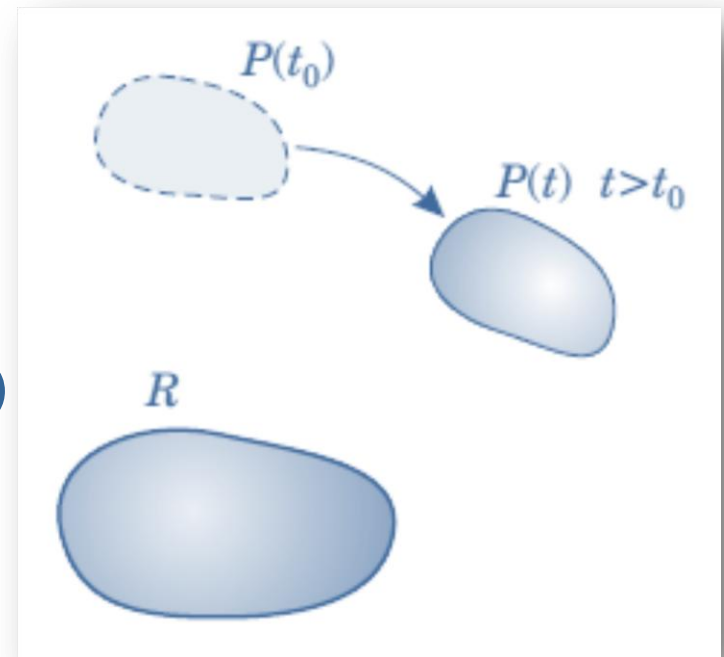
Šta ćemo naučiti?

1. Osnovni pojmovi mehanike - kinematike (Mehaničko kretanje i mirovanje. Prostor i vreme. Objekti proučavanja u mehanici. Osnovna kretanja. Broj stepeni slobode kretanja)
2. Vektor položaja tačke i trajektorija. Vektor brzine tačke. Vektor ubrzanja tačke. Ubrzano i usporeno kretanje
3. Kinematika tačke - Dekartov koordinatni sistem
4. Pravolinijsko kretanje tačke. Ravnomerno i ravnomerno promenljivo kretanje

1. Osnovni pojmovi mehanike - kinematike (Mehaničko kretanje i mirovanje. Prostor i vreme. Objekti proučavanja u mehanici. Osnovna kretanja. Broj stepeni slobode kretanja)

Mehaničko kretanje i mirovanje

- Pod mehaničkim kretanjem podrazumeva se promena položaja jednog tela u odnosu na drugo
- Posmatrano telo
- Referentno telo
- Referentna tačka
- Apsolutno kretanje (R miruje)
- Relativno kretanje (R se kreće)



Prostor i vreme

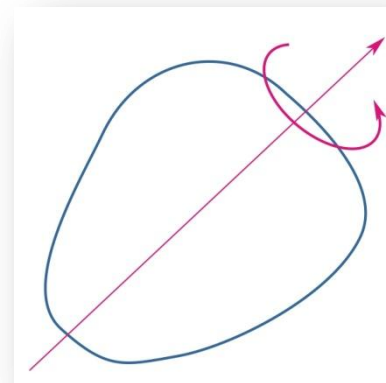
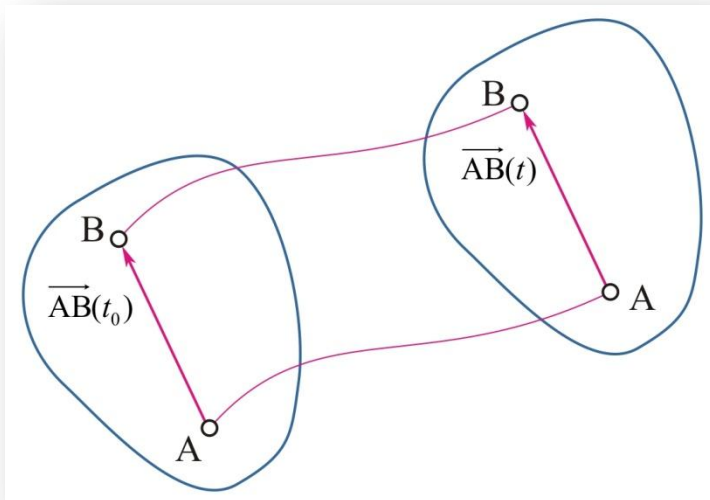
- Euklidov prostor. Jedinica za dužinu je metar [m] .
- Vreme – apsolutno ili univerzalno i nezavisno od kretanja tela. Jedinica – sekunda [s] . Realna pozitivna veličina. Proizvoljan trenutak vremena – t , početni trenutak vremena t_0 , određeni (specificirani trenutak vremena – $t_1, t_2, t^*, \dots$

Objekti proučavanja u mehanici

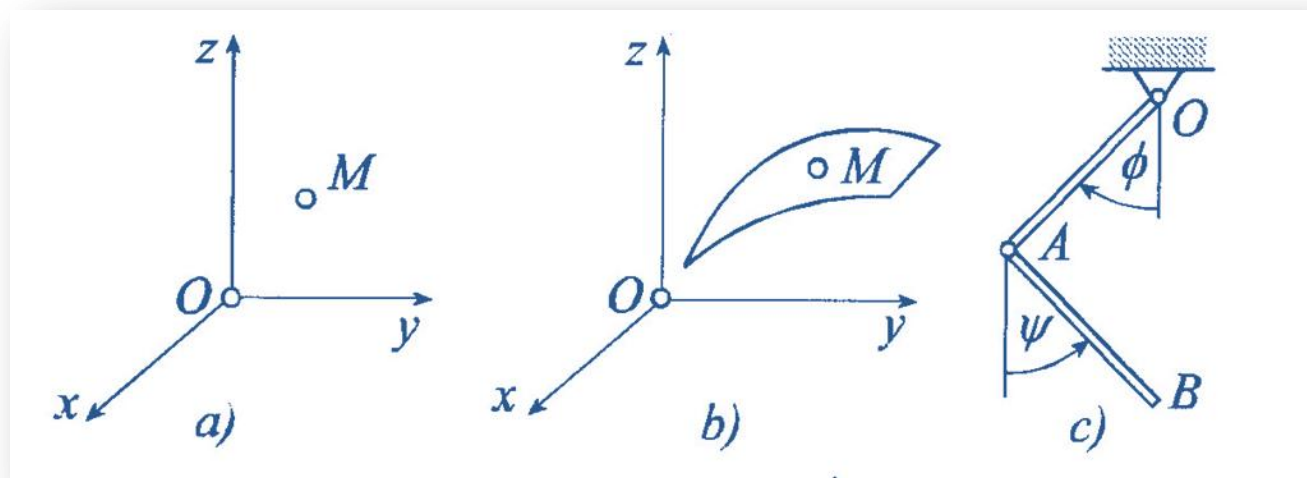
- Materijalna tačka
- Kruto telo
- Sistem materijalnih tačaka
- Sistem krutih tela

Osnovna kretanja

- Translatorno kretanje
- Obrtanje oko nepokretne ose



Broj stepeni slobode kretanja



3 st.sl.

2 st.sl.

2 st.sl

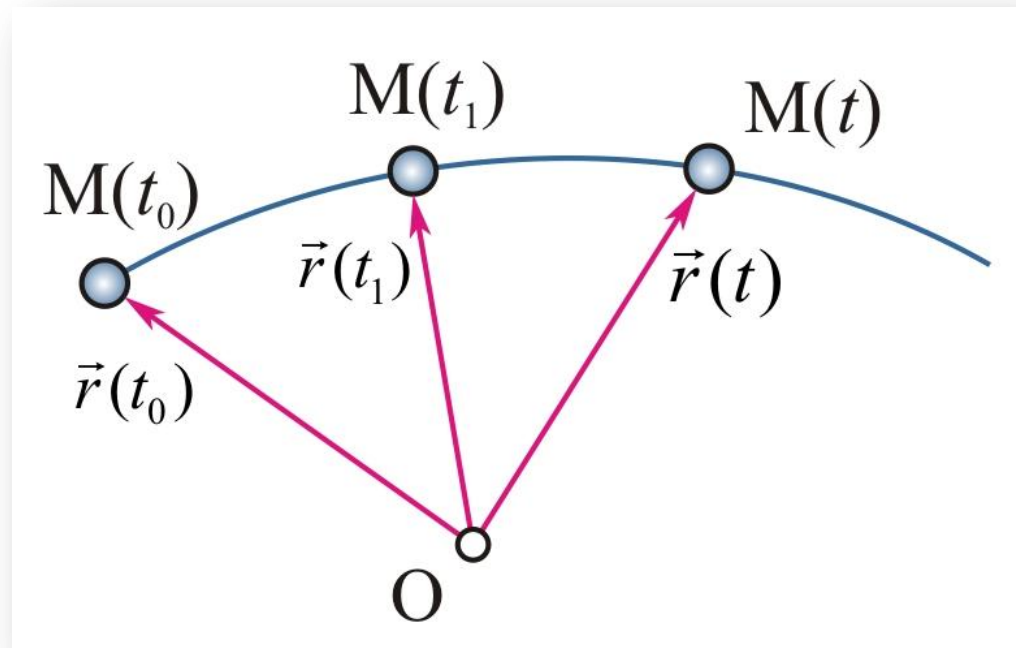
2. Vektor položaja tačke i trajektorija. Vektor brzine tačke. Vektor ubrzanja tačke

Vektor položaja tačke i trajektorija

Vektor položaja tačke

$$\vec{r} = \vec{r}(t)$$

Trajektorija – geometrijsko mesto tačaka koje opisuje vrh vektora položaja pri kretanju tačke



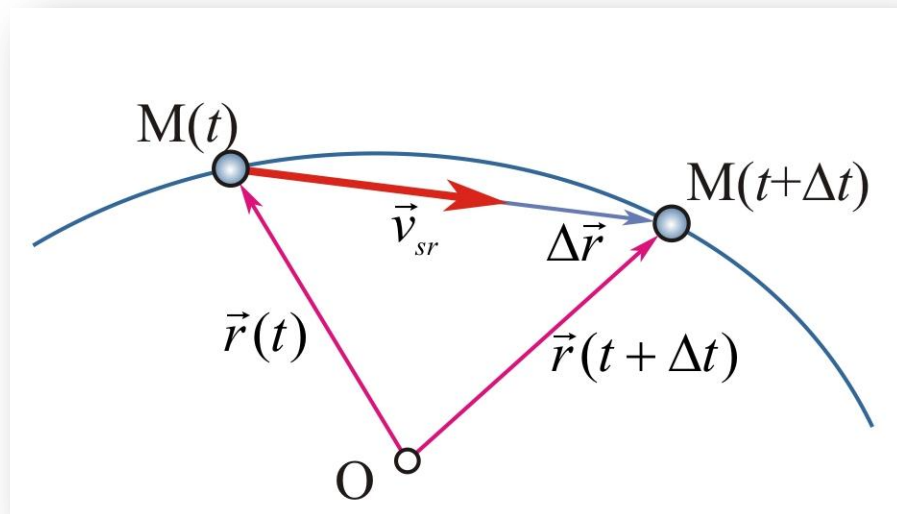
Označavanje:

$$\mathbf{r}(t) = \vec{r}(t)$$

Vektor brzine tačke

- Srednja brzina

$$\vec{v}_{sr} = \frac{\Delta \vec{r}}{\Delta t}$$



$$\Delta \vec{r} = \vec{r}(t + \Delta t) - \vec{r}(t)$$

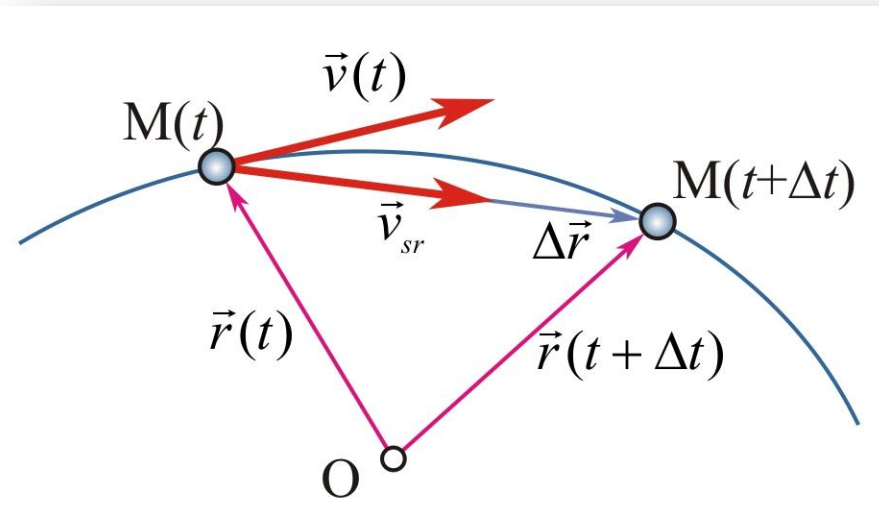
Jedinica – [m/s]

Vektor brzine tačke

- Trenutna brzina

$$\vec{v}(t) = \lim_{\Delta t \rightarrow 0} \vec{v}_{sr} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \dot{\vec{r}}(t)$$



Brzina tačke je prvi izvod vektora položaja po vremenu.

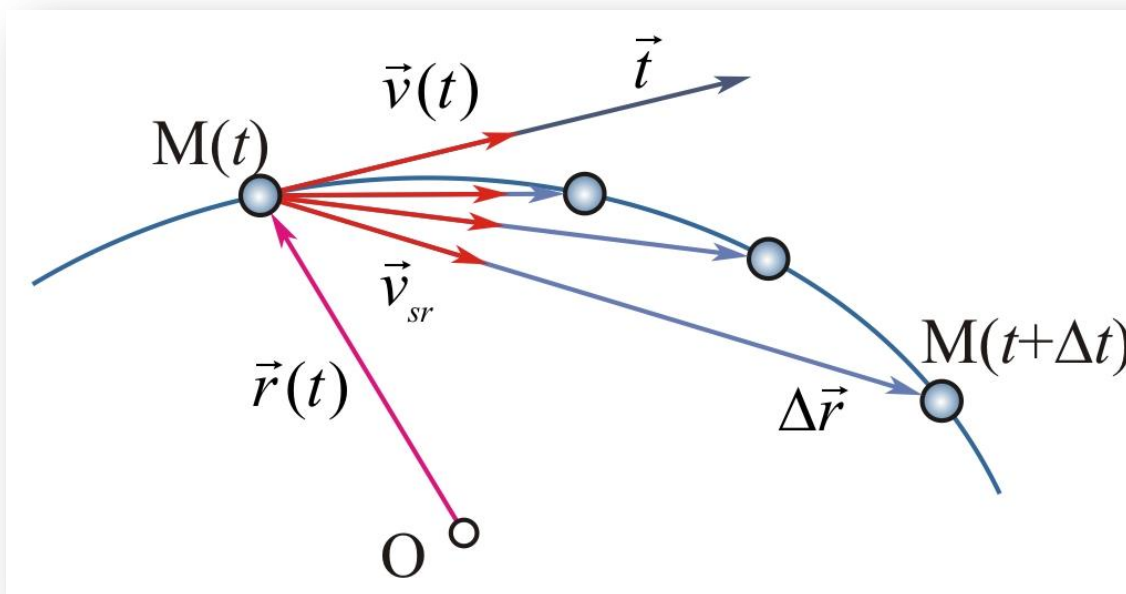
Vektor brzine tačke

- Srednja brzina

$$\vec{v}_{sr} = \frac{\Delta \vec{r}}{\Delta t}$$

- Trenutna brzina

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \dot{\vec{r}}(t)$$

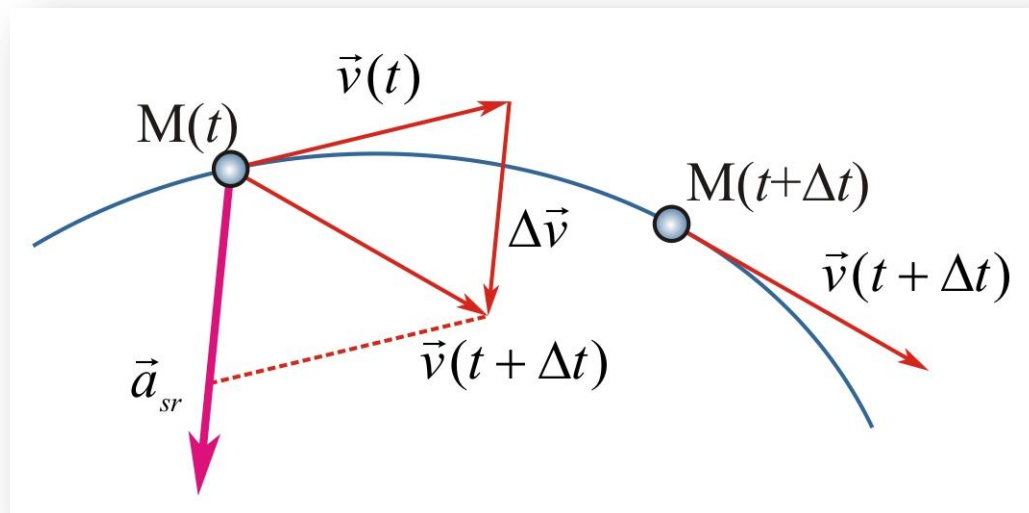


Vektor brzine tačke ima pravac tangente na trajektoriju.

Vektor ubrzanja tačke

- Srednje ubrzanje

$$\vec{a}_{sr} = \frac{\Delta \vec{v}}{\Delta t}$$



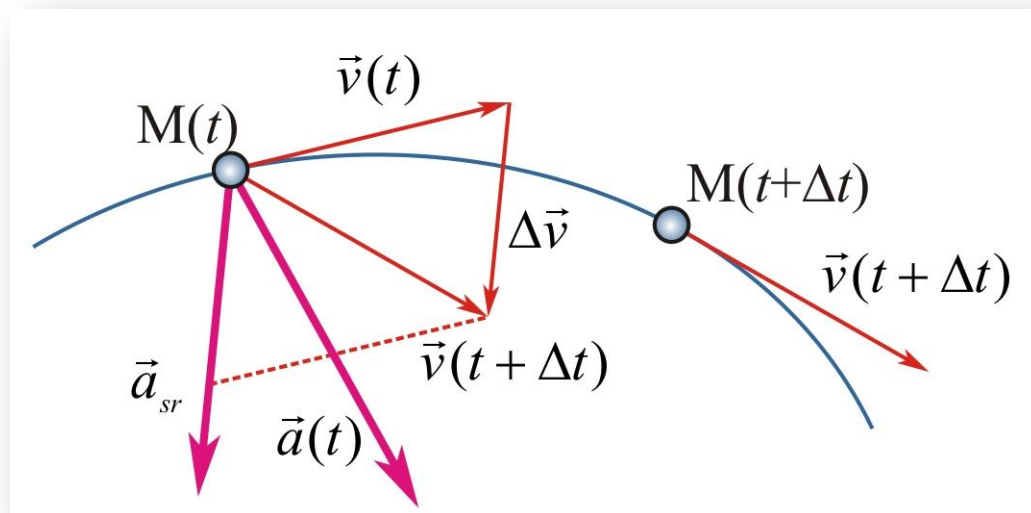
$$\Delta \vec{v} = \vec{v}(t + \Delta t) - \vec{v}(t)$$

Jedinica – $[m/s^2]$

Vektor ubrzanja tačke

- Trenutno ubrzanje

$$\begin{aligned}\vec{a}(t) &= \lim_{\Delta t \rightarrow 0} \vec{a}_{sr} \\ &= \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}\end{aligned}$$



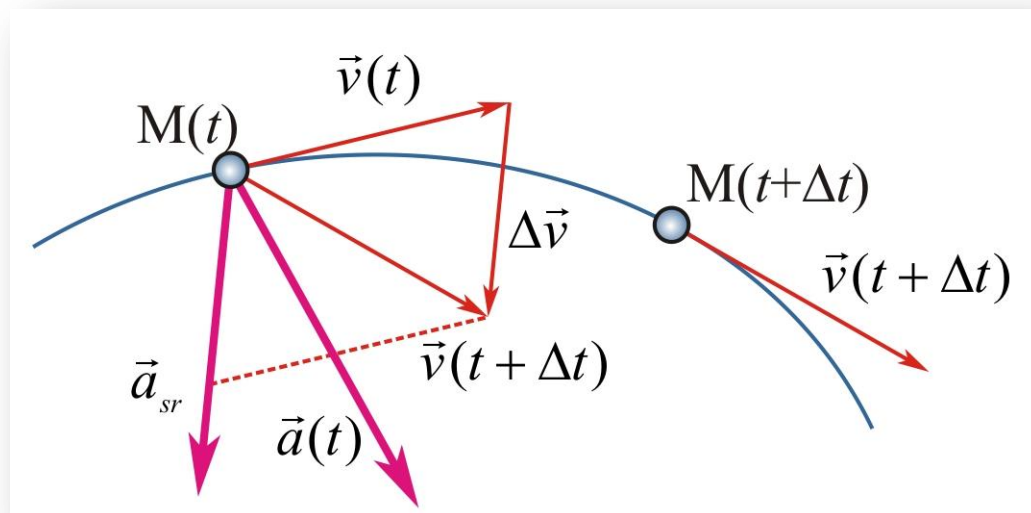
$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \dot{\vec{v}}(t) = \frac{d^2\vec{r}}{dt^2} = \ddot{\vec{r}}(t)$$

Ubrzanje tačke je prvi izvod vektora brzine, odnosno, drugi izvod vektora položaja po vremenu.

Vektor ubrzanja tačke

- Trenutno ubrzanje

$$\begin{aligned}\vec{a}(t) &= \lim_{\Delta t \rightarrow 0} \vec{a}_{sr} \\ &= \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}\end{aligned}$$



$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \dot{\vec{v}}(t) = \frac{d^2\vec{r}}{dt^2} = \ddot{\vec{r}}(t)$$

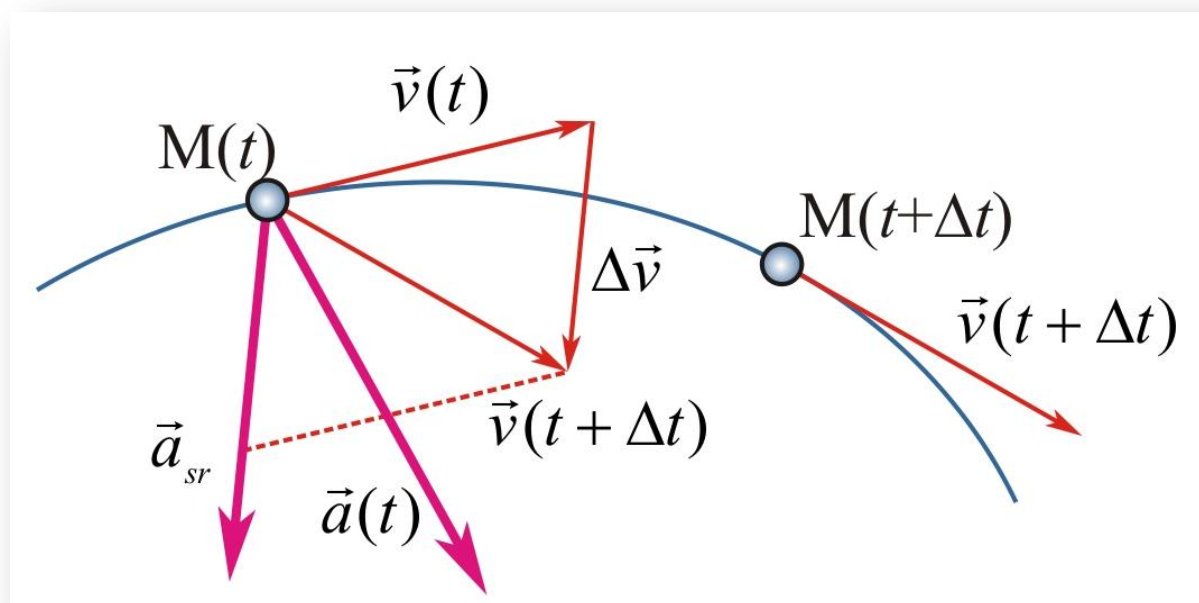
Vektor ubrzanja tačke pada u konkavnu (izdubljenu) stranu trajektorije kretanja tačke.

Vektor ubrzanja tačke

- Srednje ubrzanje
- Trenutno ubrzanje

$$\vec{a}_{sr} = \frac{\Delta \vec{v}}{\Delta t}$$

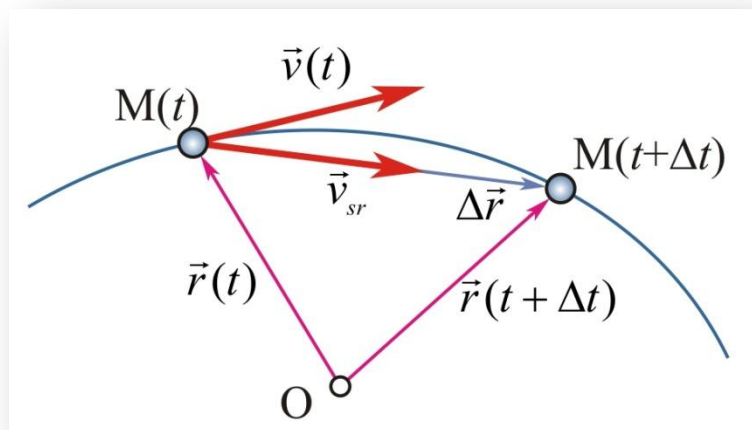
$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \dot{\vec{v}}(t) = \frac{d^2\vec{r}}{dt^2} = \ddot{\vec{r}}(t)$$



Brzina i ubrzanje tačke

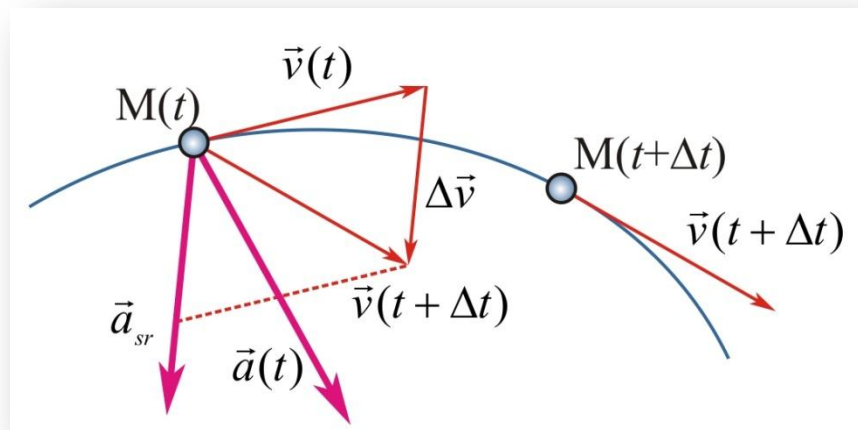
- Brzina

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \dot{\vec{r}}(t)$$



- Ubrzanje

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \dot{\vec{v}}(t) = \frac{d^2\vec{r}}{dt^2} = \ddot{\vec{r}}(t)$$



Ubrzano i usporeno kretanje

Ubrzano kretanje

$$|\vec{v}| = v \quad \nearrow$$

$$v^2 \nearrow$$

$$v^2 = \vec{v} \cdot \vec{v} \quad \nearrow$$

$$\frac{d}{dt}(v^2) = \frac{d}{dt}(\underbrace{\vec{v} \cdot \vec{v}}_{> 0}) > 0$$

$$\vec{a} \cdot \vec{v} + \vec{v} \cdot \vec{v} > 0$$

$$\underbrace{\vec{a} \cdot \vec{v} > 0}$$

$$\vec{a} \cdot \vec{v} = a \cdot v \cdot \cos \phi(\vec{a}, \vec{v}) > 0$$

$$\cos \phi(\vec{a}, \vec{v}) > 0$$

Usporeno kretanje

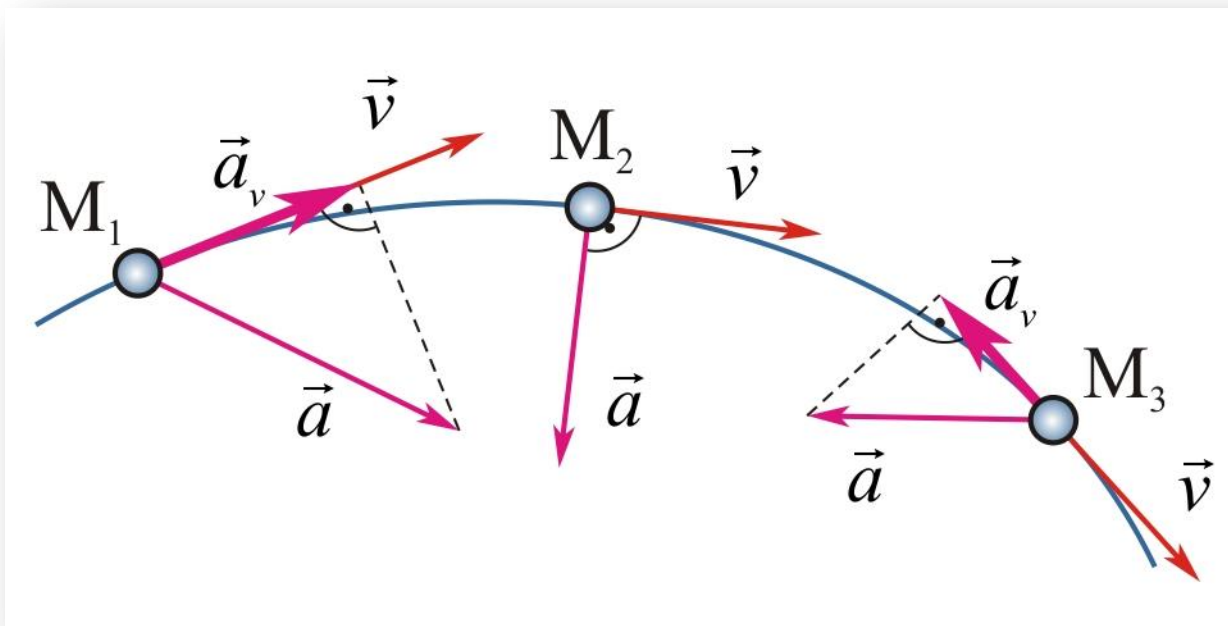
$$|\vec{v}| = v \quad \searrow$$

$$\vdots$$

$$\vec{a} \cdot \vec{v} < 0$$

$$\cos \phi(\vec{a}, \vec{v}) < 0$$

Ubrzano i usporeno kretanje



Ubrzano kretanje

$$\vec{a} \cdot \vec{v} > 0$$

Usporeno kretanje

$$\vec{a} \cdot \vec{v} < 0$$

3. Kinematika tačke - Dekartov koordinatni sistem

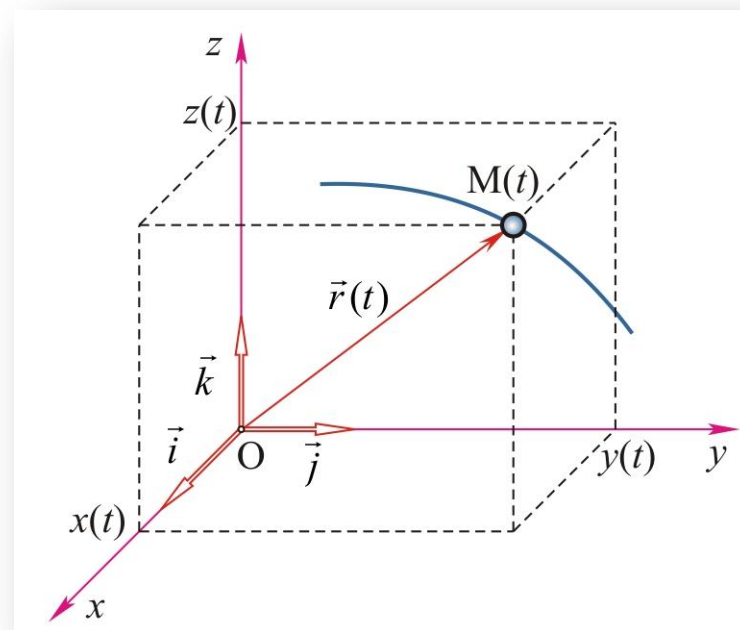
Dekartov koordinatni sistem – vektor položaja

- Vektor položaja

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$$

- Parametarske jednačine kretanja:

$$x = x(t) \quad y = y(t) \quad z = z(t)$$



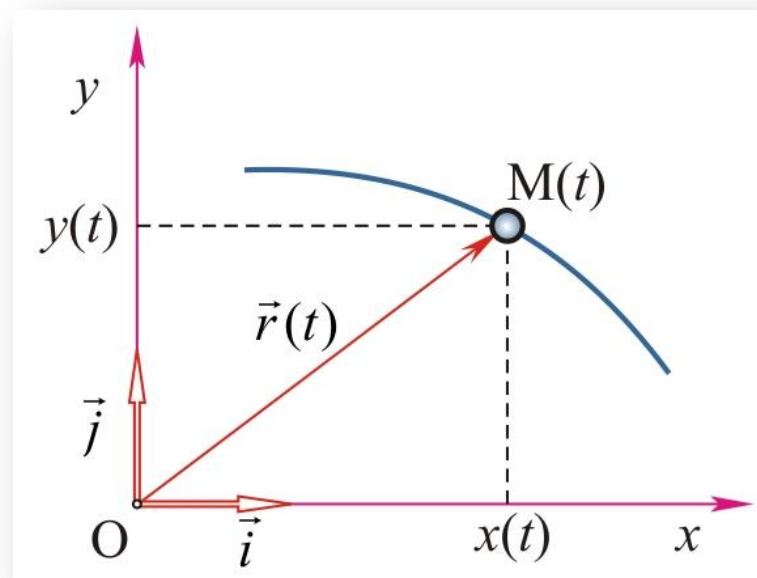
Dekartov koordinatni sistem – vektor položaja

- Vektor položaja

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$$

- Parametarske jednačine kretanja:

$$x = x(t) \quad y = y(t)$$



Dekartov koordinatni sistem – trajektorija

- Oblast kretanja

$$x_1 \leq x(t) \leq x_2$$

$$y_1 \leq y(t) \leq y_2$$

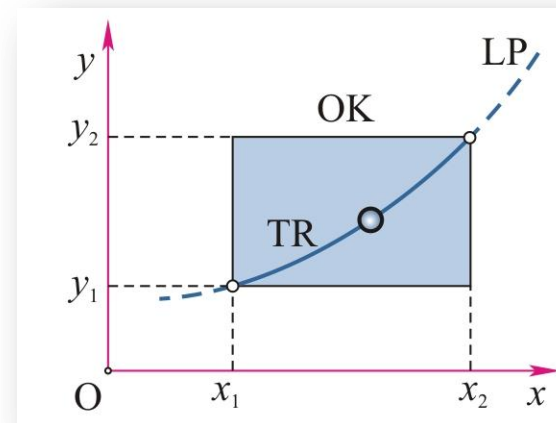
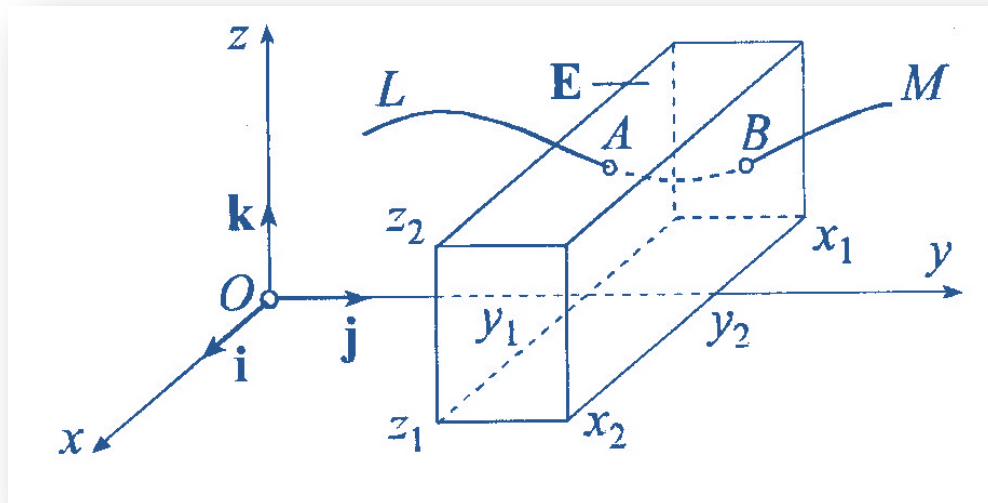
$$z_1 \leq z(t) \leq z_2$$

$$\text{za } t \geq t_0$$

- Linija putanje

$$\left. \begin{matrix} x(t) \\ y(t) \\ z(t) \end{matrix} \right\} \rightarrow \begin{cases} f_1(x, y, z) = 0 \\ f_2(x, y, z) = 0 \end{cases}$$

$$\left. \begin{matrix} x(t) \\ y(t) \end{matrix} \right\} \rightarrow y(x)$$



Dekartov koordinatni sistem – brzina

- Vektor brzine

$$\vec{v}(t) = \vec{v}_x(t) + \vec{v}_y(t)$$

$$= v_x(t)\vec{i} + v_y(t)\vec{j}$$

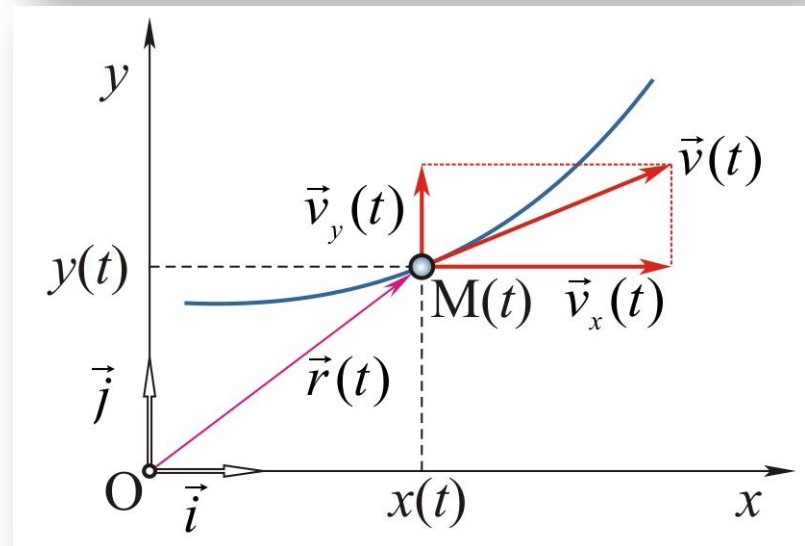
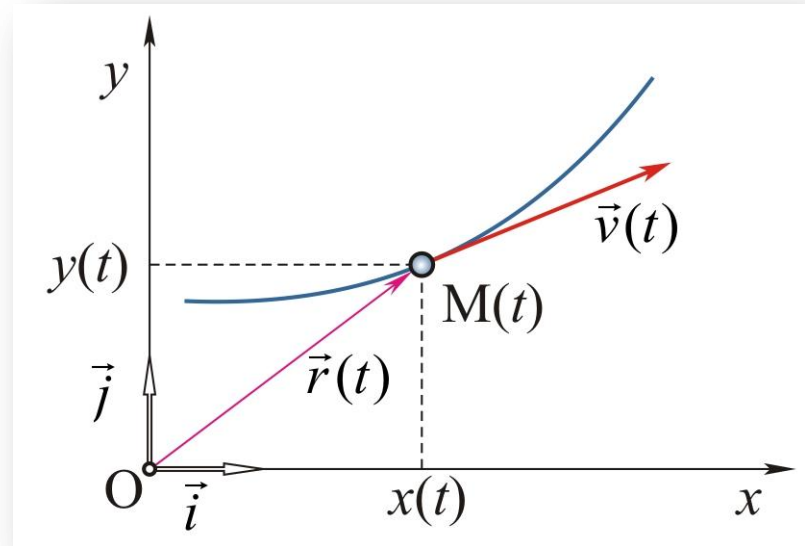
$$\vec{v}(t) = \dot{\vec{r}}(t) = \dot{x}(t)\vec{i} + \dot{y}(t)\vec{j}$$

- Projekcije vektora brzine:

$$v_x(t) = \dot{x}(t) \quad , \quad v_y(t) = \dot{y}(t)$$

- Intenzitet vektora brzine

$$|\vec{v}(t)| = v(t) = \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}$$



Dekartov koordinatni sistem – ubrzanje

- Vektor ubrzanja

$$\vec{a}(t) = \vec{a}_x(t) + \vec{a}_y(t)$$

$$= a_x(t)\vec{i} + a_y(t)\vec{j}$$

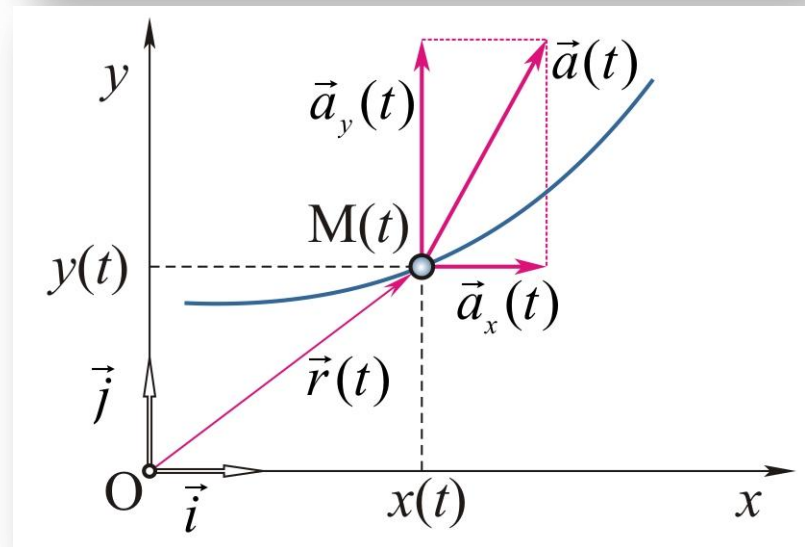
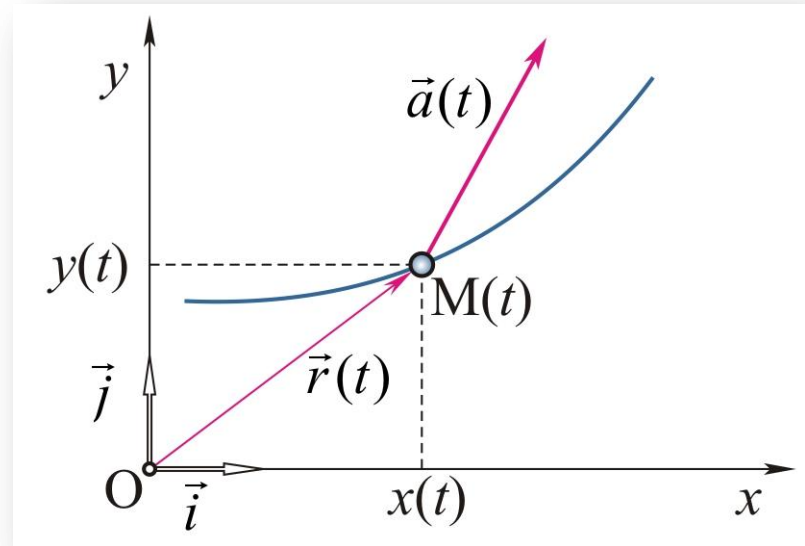
$$\vec{a}(t) = \dot{\vec{v}}(t) = \ddot{\vec{r}}(t) = \ddot{x}(t)\vec{i} + \ddot{y}(t)\vec{j}$$

- Projekcije vektora ubrzanja:

$$a_x(t) = \ddot{x}(t) \quad , \quad a_y(t) = \ddot{y}(t)$$

- Intenzitet vektora ubrzanja

$$|\vec{a}(t)| = a(t) = \sqrt{\ddot{x}(t)^2 + \ddot{y}(t)^2}$$



Dekartov koordinatni sistem

- Položaj

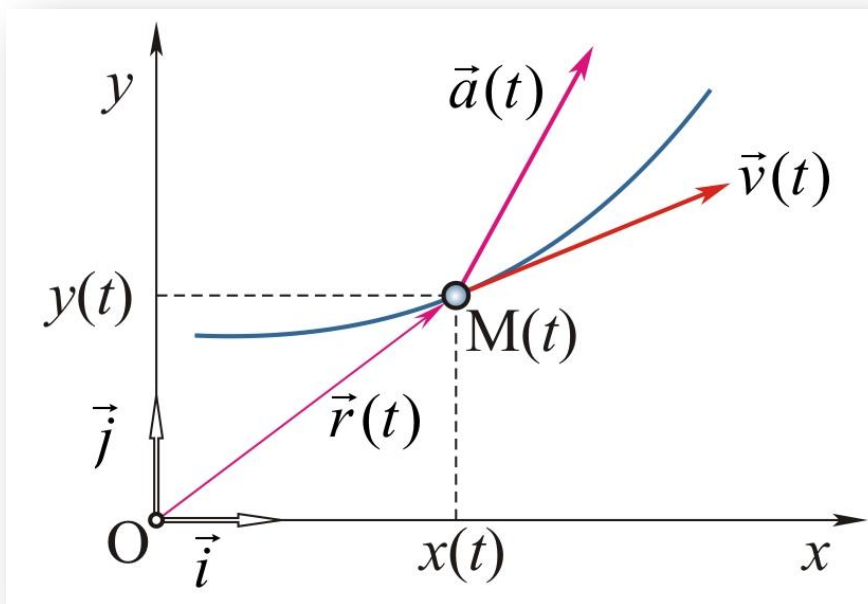
$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$$

- Brzina

$$\vec{v}(t) = \dot{x}(t)\vec{i} + \dot{y}(t)\vec{j}$$

- Ubrzanje

$$\vec{a}(t) = \ddot{x}(t)\vec{i} + \ddot{y}(t)\vec{j}$$



Primer 1

Kretanje tačke je opisano parametarskim jednačinama

$$x(t) = \frac{t^2}{2} - t \quad y(t) = \frac{t}{2}$$

- a) Odrediti trajektoriju tačke,
- b) odrediti trenutak $t^* > 0$ u kome će se tačka naći na osi y ,
- c) odrediti brzinu i ubrzanje tačke u proizvoljnom trenutku vremena t ,
- d) odrediti brzinu i ubrzanje tačke, i njihove intenzitete, u trenutku t^* .

a) trajektorija

$$\left. \begin{array}{l} x = \frac{t^2}{2} - t \\ y = \frac{t}{2} \end{array} \right\} \rightarrow \left. \begin{array}{l} x = \frac{t^2}{2} - t \\ t = 2y \end{array} \right\} \rightarrow x = \frac{1}{2}(2y)^2 - (2y) \rightarrow x = 2y^2 - 2y$$

$$y(t) = \frac{t}{2} \quad t \geq 0 \rightarrow y \geq 0$$

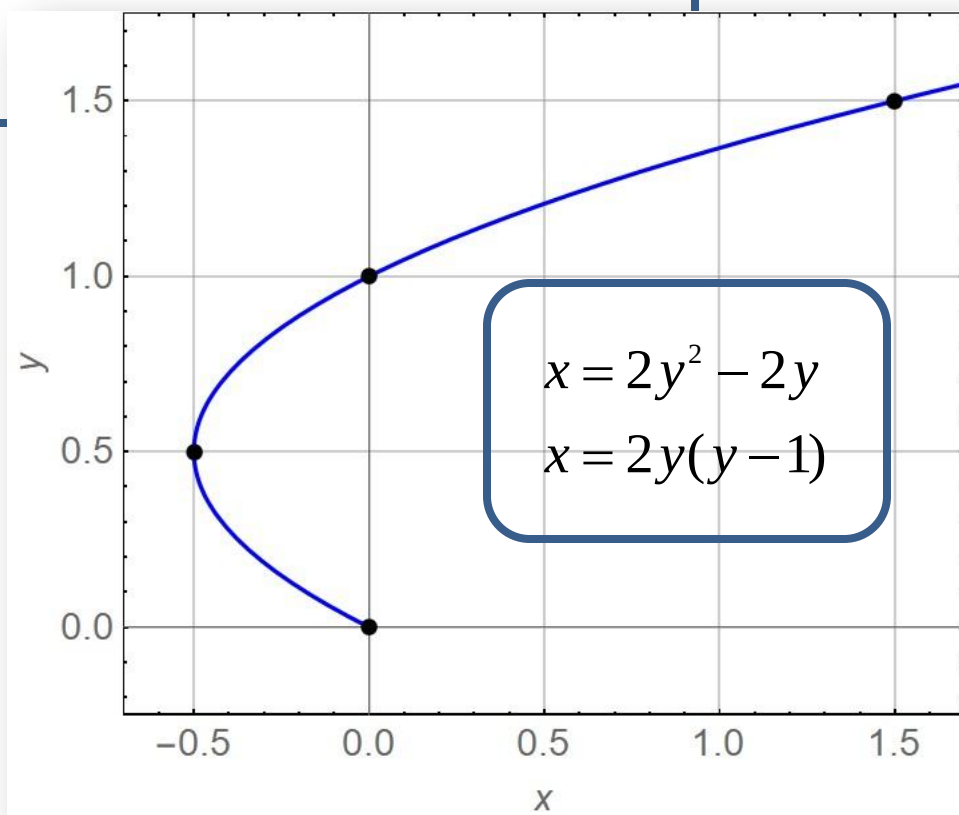
$$\underbrace{x(t) = \frac{t^2}{2} - t \quad y(t) = \frac{t}{2}}_{\downarrow}$$

$$x(0) = 0 \quad y(0) = 0$$

$$x(1) = -\frac{1}{2} \quad y(1) = \frac{1}{2}$$

$$x(2) = 0 \quad y(2) = 1$$

$$x(3) = \frac{3}{2} \quad y(3) = \frac{3}{2}$$

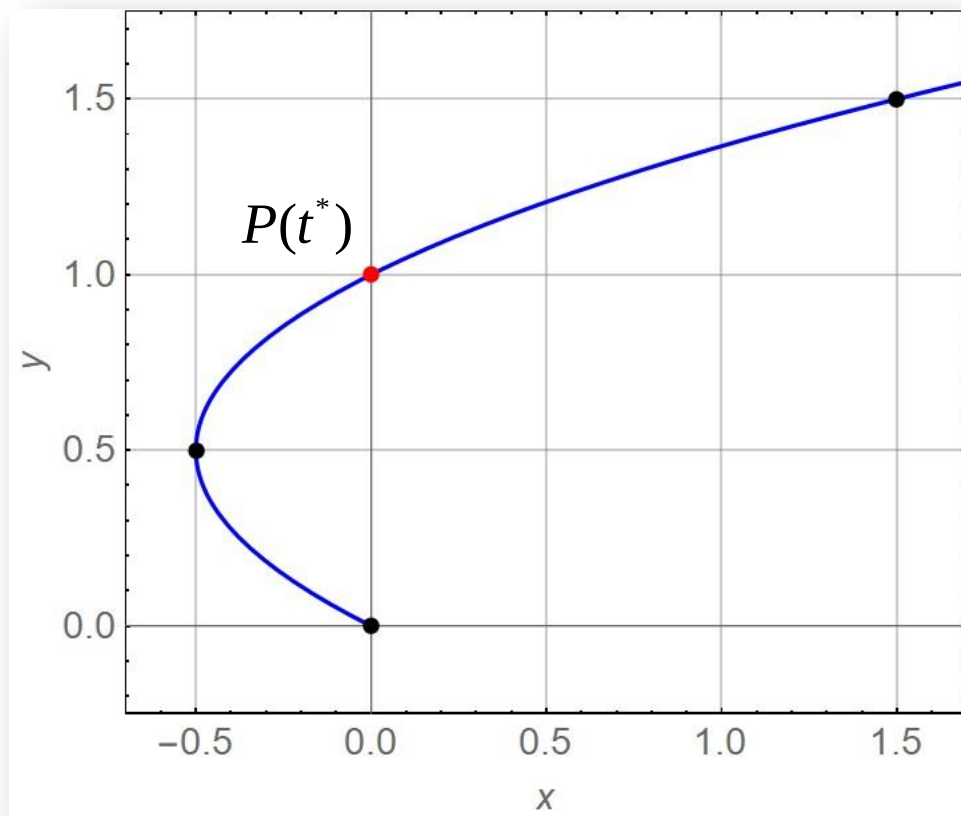


b) trenutak $t^* > 0$ yu kome će se tačka naći na osi y

$$\begin{aligned}x(t^*) &= 0 \\ \frac{(t^*)^2}{2} - t^* &= 0 \\ t^* \left(\frac{t^*}{2} - 1 \right) &= 0 \\ t^* &= 2\end{aligned}$$

$$\underbrace{x(t) = \frac{t^2}{2} - t \quad y(t) = \frac{t}{2}}_{\downarrow}$$

$$x(t^*) = x(2) = 0 \quad y(t^*) = y(2) = 1$$



c) brzina i ubrzanje

$$\underbrace{x(t) = \frac{t^2}{2} - t \quad y(t) = \frac{t}{2}}$$

$$\dot{x}(t) = t - 1 \quad \dot{y}(t) = \frac{1}{2}$$

$$\ddot{x}(t) = 1 \quad \ddot{y}(t) = 0$$

$$\vec{a}(t) = \ddot{x}(t)\vec{i} + \ddot{y}(t)\vec{j}$$

$$\vec{a}(t) = 1\vec{i} + 0\vec{j}$$

$$|\vec{a}(t)| = a(t) = \sqrt{\ddot{x}(t)^2 + \ddot{y}(t)^2}$$

$$a(t) = \sqrt{1^2 + 0^2} = 1$$

$$\vec{v}(t) = \dot{x}(t)\vec{i} + \dot{y}(t)\vec{j}$$

$$\vec{v}(t) = (t - 1)\vec{i} + \frac{1}{2}\vec{j}$$

$$|\vec{v}(t)| = v(t) = \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}$$

$$v(t) = \sqrt{(t - 1)^2 + \left(\frac{1}{2}\right)^2}$$

d) brzina i ubrzanje u trenutku t^*

$$t^* = 2$$

$$\vec{v}(t) = (t-1)\vec{i} + \frac{1}{2}\vec{j} \rightarrow \vec{v}(2) = 1\vec{i} + \frac{1}{2}\vec{j}$$

$$v(t) = \sqrt{(t-1)^2 + \left(\frac{1}{2}\right)^2} \rightarrow v(2) = \frac{\sqrt{5}}{2}$$

$$\vec{a}(t) = 1\vec{i} \rightarrow \vec{a}(2) = 1\vec{i}$$

$$a(t) = 1 \rightarrow a(2) = 1$$

d) brzina i ubrzanje u trenutku t^*

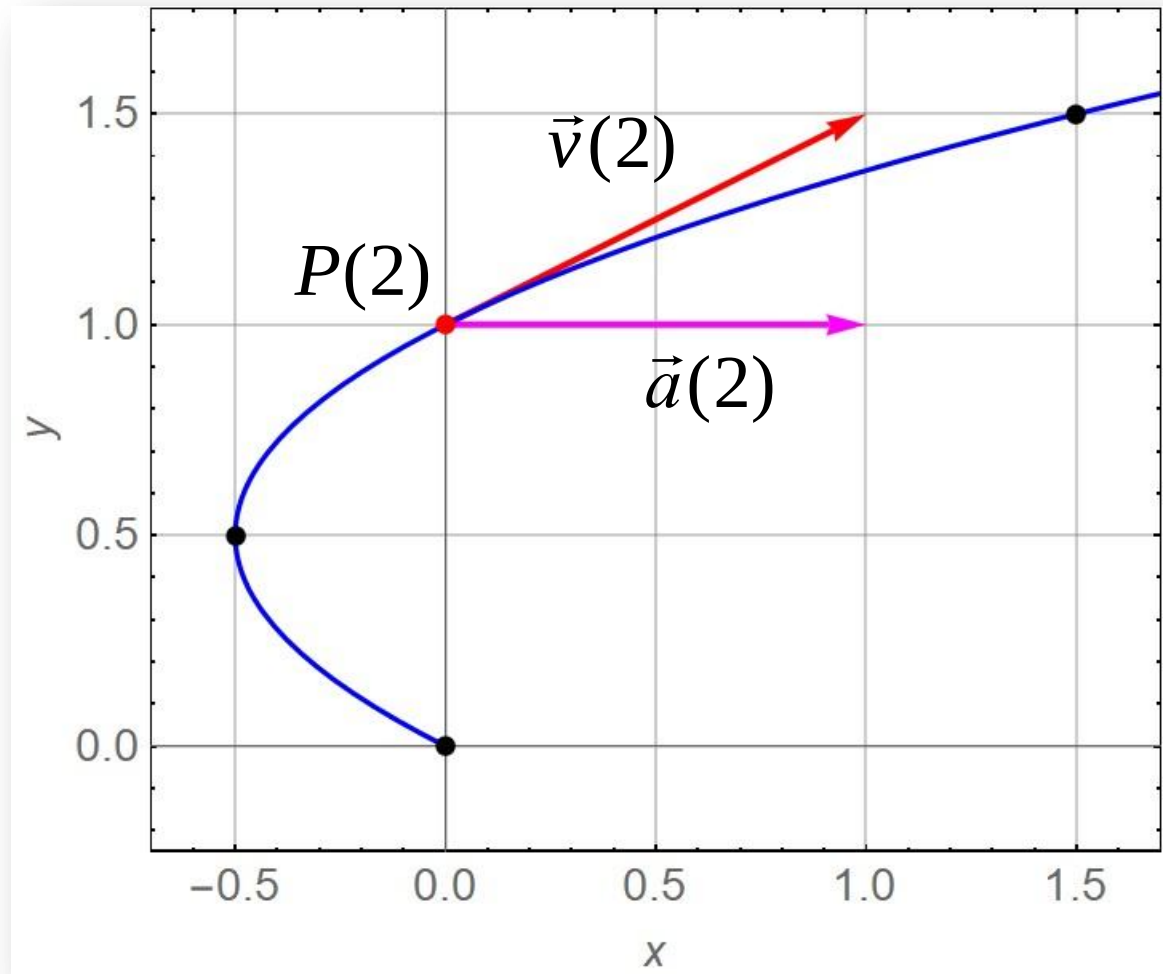
$$t^* = 2$$

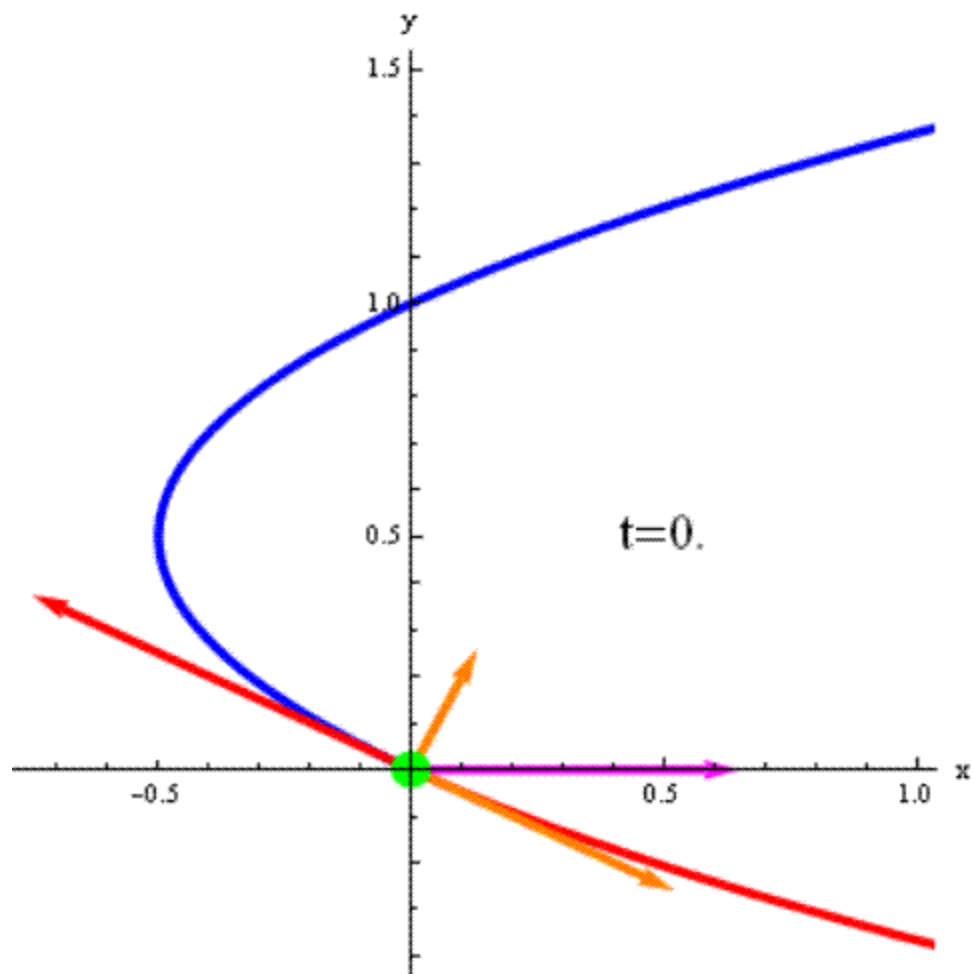
$$\vec{v}(2) = 1\vec{i} + \frac{1}{2}\vec{j}$$

$$v(2) = \frac{\sqrt{5}}{4}$$

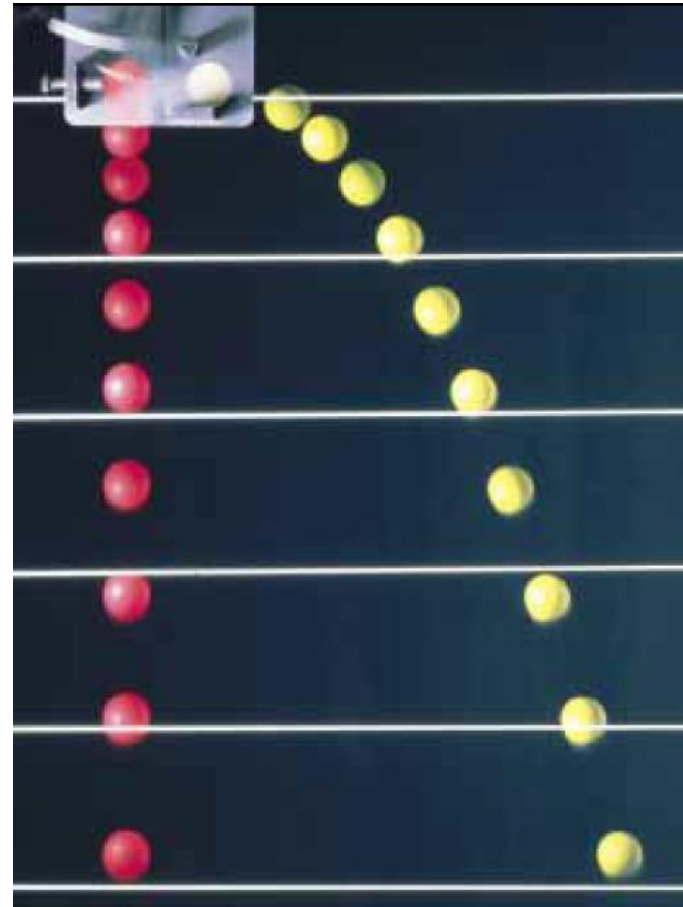
$$\vec{a}(2) = 1\vec{i}$$

$$a(2) = 1$$





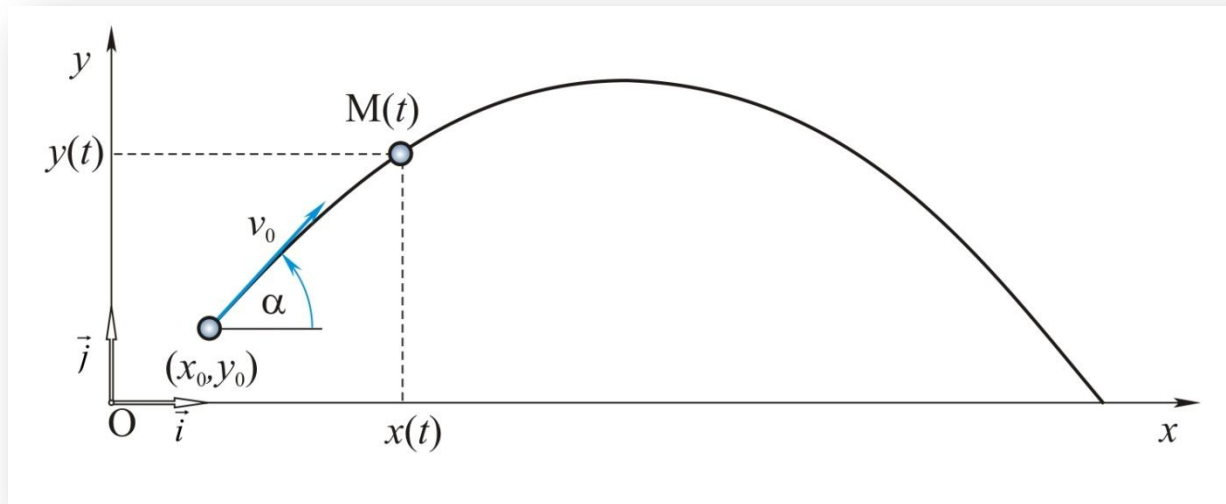
Primer 2 – kosi hitac



35. Kosi hitac



<http://www.mech-in-ns.ftn.uns.ac.rs/projects/35-kosi-hitac/>



- Parametarske jednačine kretanja

$$x(t) = v_0 t \cos \alpha + x_0$$

$$y(t) = -\frac{1}{2} g t^2 + v_0 t \sin \alpha + y_0$$

- trajektorija ($x_0=0, y_0=0$)

$$\dot{x}(t) = v_0 \cos \alpha$$

$$\ddot{x}(t) = 0$$

$$\dot{y}(t) = -gt + v_0 \sin \alpha$$

$$\ddot{y}(t) = -g$$

$$y(x) = x \operatorname{tg} \alpha - \frac{g}{2 v_0^2 \cos^2 \alpha} x^2$$

- domet

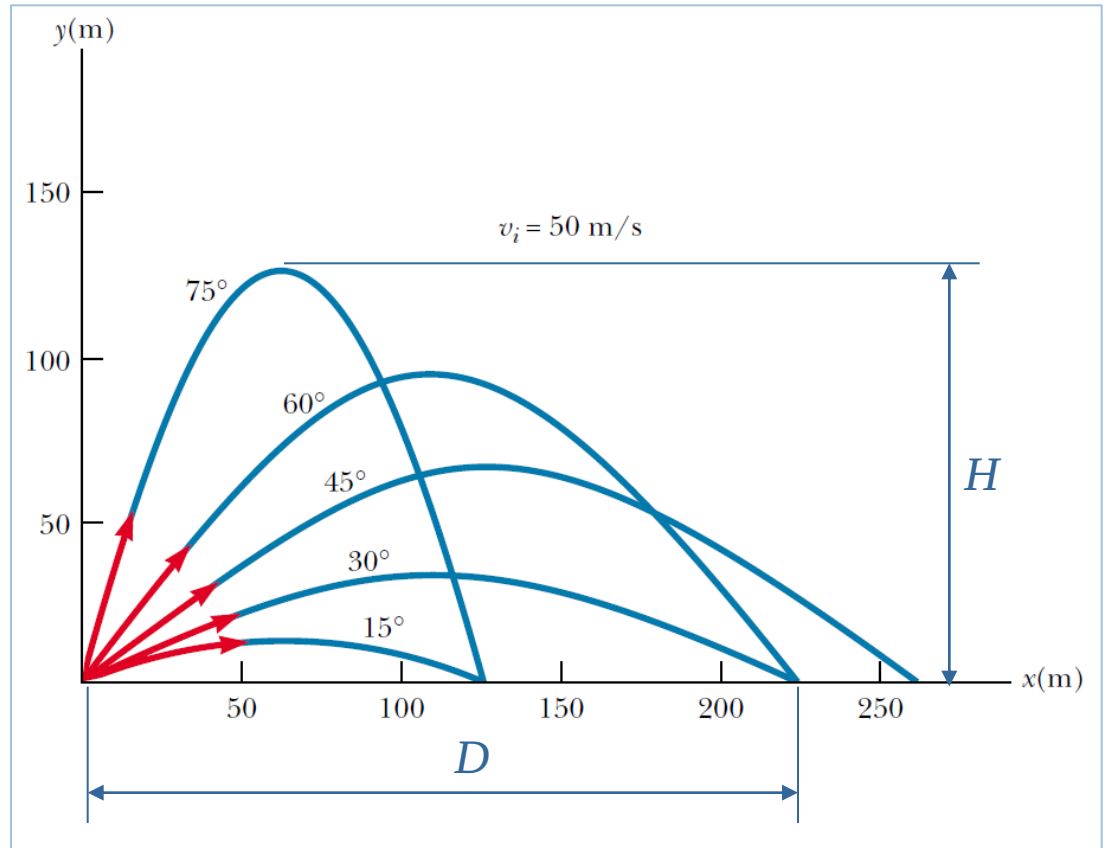
$$D = \frac{v_0^2 \sin 2\alpha}{g}$$

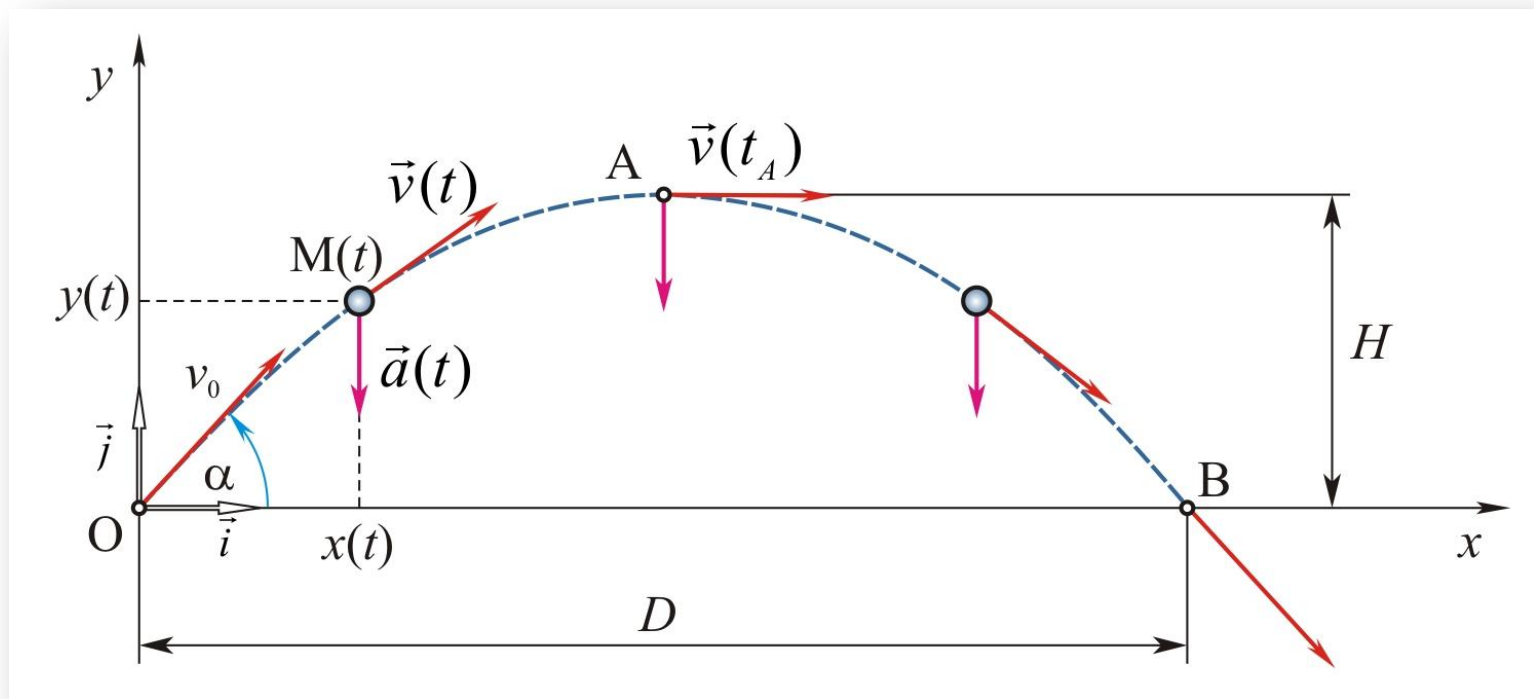
$$D_{\max} = \frac{v_0^2}{g}, \text{ za } \alpha = 45^\circ$$

- visina leta

$$H = \frac{v_0^2 \sin^2 \alpha}{2g}$$

$$H_{\max} = \frac{v_0^2}{2g}, \text{ za } \alpha = 90^\circ$$





4. Pravolinijsko kretanje tačke. Ravnomerno i ravnomerno promenljivo kretanje

Pravolinijsko kretanje tačke

- Vektor položaja

$$\vec{r}(t) = x(t)\vec{i}$$

- Brzina

$$\vec{v}(t) = v_x(t)\vec{i} = \dot{x}(t)\vec{i} = \dot{\vec{r}}(t)$$

- Ubrzanje

$$\vec{a}(t) = a_x(t)\vec{i} = \dot{v}_x(t)\vec{i} = \ddot{x}(t)\vec{i} = \dot{\vec{v}}(t) = \ddot{\vec{r}}(t)$$

Primer 3

- Pravolinijsko kretanje tačke opisano je parametarskom jednačinom

$$x(t) = 6t^2 - t^3$$

- Odrediti brzinu i ubrzanje tačke i nacrtati dijagrame njihove promene,
- odrediti trenutak t^* u kom tačka menja smer kretanja,
- odrediti put koji će tačka preći tokom prvih 6s kretanja,
- odrediti intervale vremena, tokom prvih 6s kretanja, u kojima će se tačka kretati ubrzano, odnosno, usporeno.

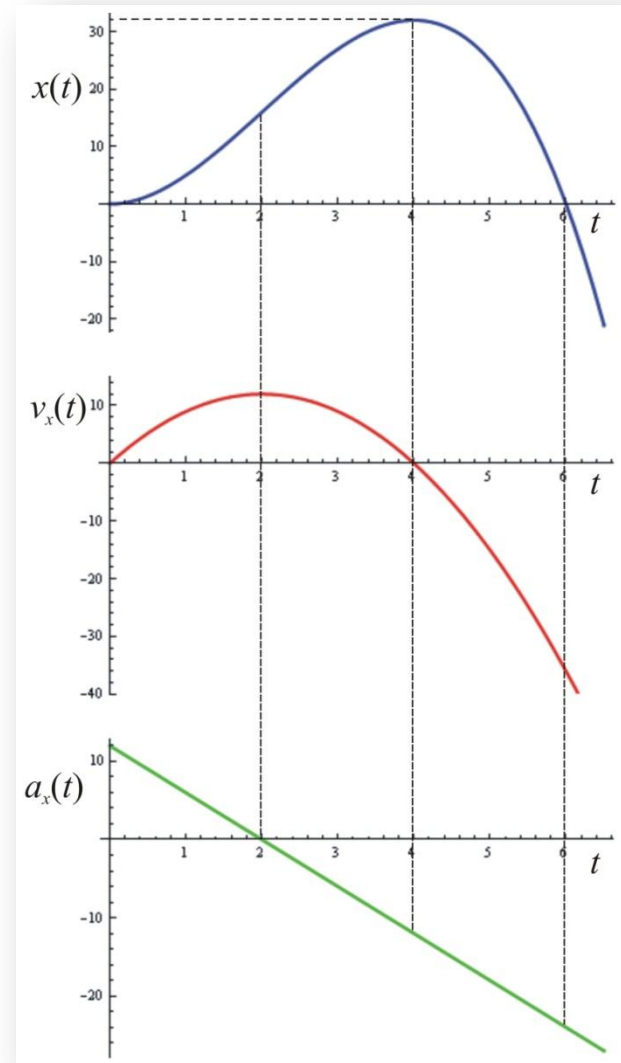
$$x(t) = 6t^2 - t^3 = t^2(6 - t)$$

↓

$$\dot{x}(t) = 12t - 3t^2 = 3t(4 - t)$$

↓

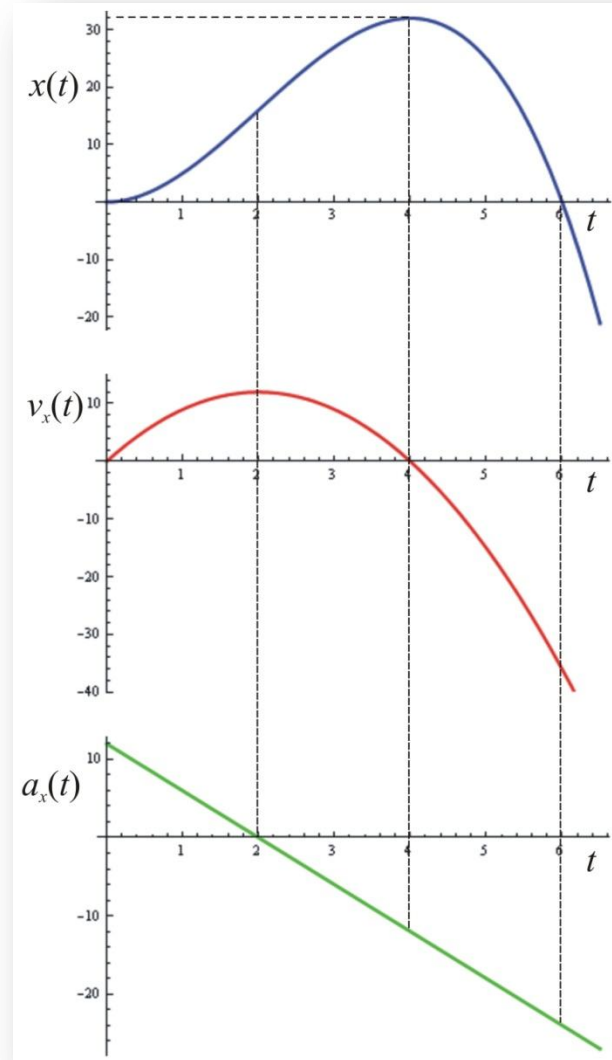
$$\ddot{x}(t) = 12 - 6t = 6(2 - t)$$



$$\dot{x}(t^*) = 3t^*(4 - t^*) = 0$$

↓

$$t^* = 4$$



$$P[0,6] = P[0,4] + P[4,6]$$

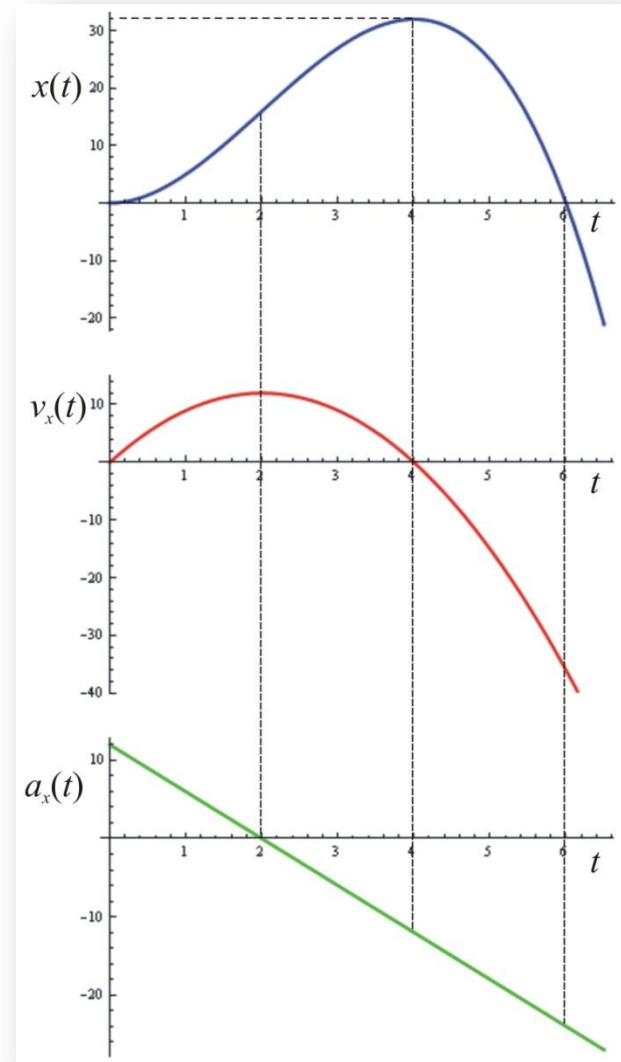
↓

$$P[0,6] = |x(4) - x(0)| + |x(6) - x(4)|$$

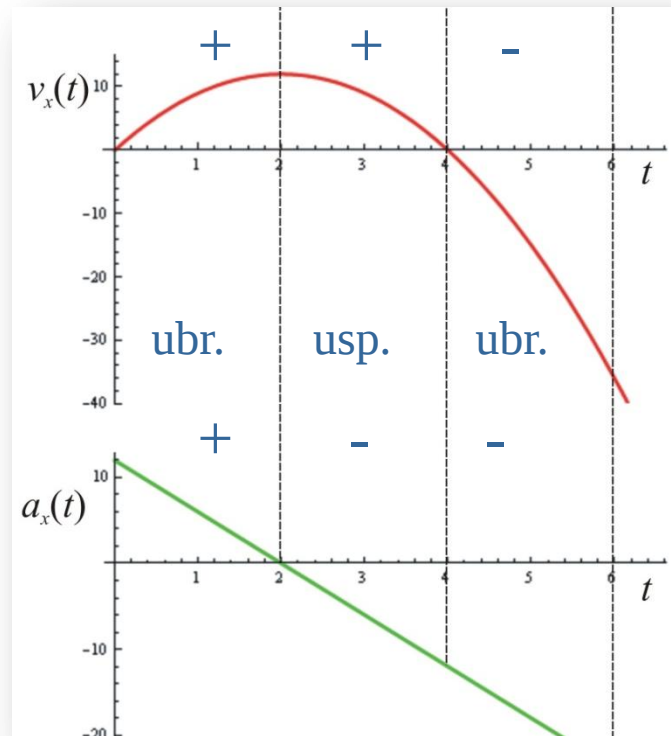
↓

$$P[0,6] = |32 - 0| + |0 - 32| = 64$$

$$x(t) = 6t^2 - t^3 = t^2(6 - t)$$



$t \in (0,2)$, $\dot{x}(t) > 0, \ddot{x}(t) > 0 \rightarrow$ ubrzano kretanje
 $t \in (0,2)$, $\dot{x}(t) > 0, \ddot{x}(t) < 0 \rightarrow$ usporeno kretanje
 $t \in (4,6)$, $\dot{x}(t) < 0, \ddot{x}(t) < 0 \rightarrow$ ubrzano kretanje

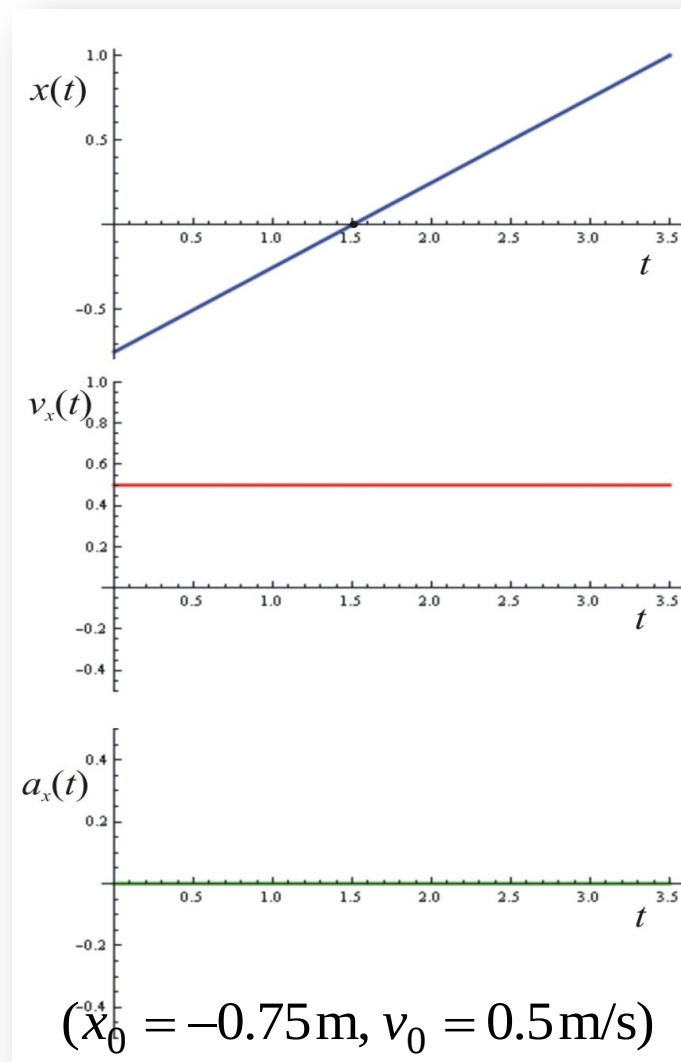


Ravnomerno kretanje tačke

$$v_x(t) = v_0 = \text{const.}$$

$$a_x(t) = 0 = \text{const.}$$

$$x(t) = v_0 t + x_0$$



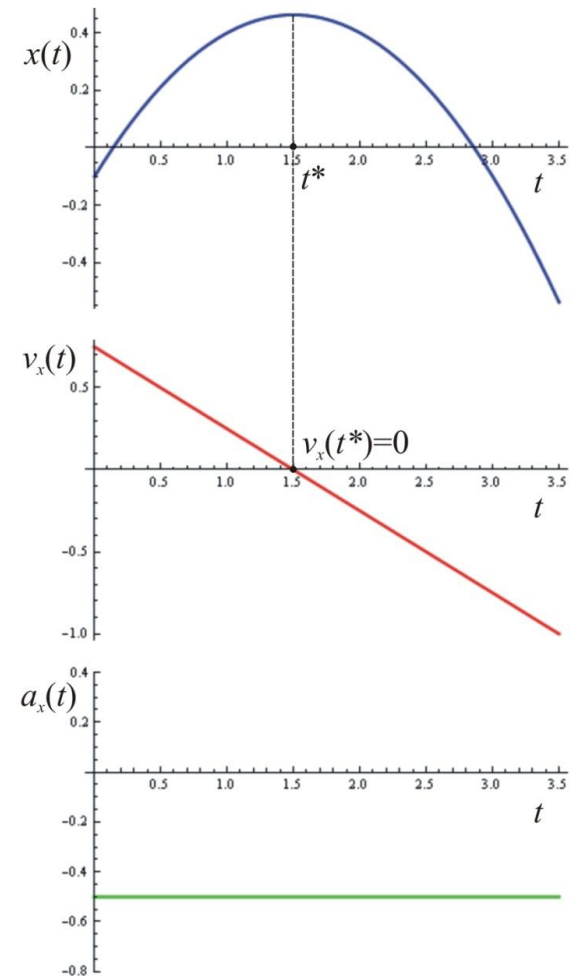
Ravnomerno kretanje tačke

Ravnomerno promenljivo kretanje tačke

$$a_x(t) = a_0 = \text{const}$$

$$v_x(t) = a_0 t + v_0$$

$$x(t) = \frac{1}{2} a_0 t^2 + v_0 t + x_0$$



Ravnomerno promenljivo kretanje tačke

Šta smo naučili?

1. Osnovni pojmovi mehanike - kinematike (Mehaničko kretanje i mirovanje. Prostor i vreme. Objekti proučavanja u mehanici. Osnovna kretanja. Broj stepeni slobode kretanja)
2. Vektor položaja tačke i trajektorija. Vektor brzine tačke. Vektor ubrzanja tačke. Ubrzano i usporeno kretanje
3. Kinematika tačke - Dekartov koordinatni sistem
4. Pravolinijsko kretanje tačke. Ravnomerno i ravnomerno promenljivo kretanje

Kinematika

Kinematika tačke – 1. deo

Kinematika i dinamika

Miodrag Zuković

Novi Sad, 2021.