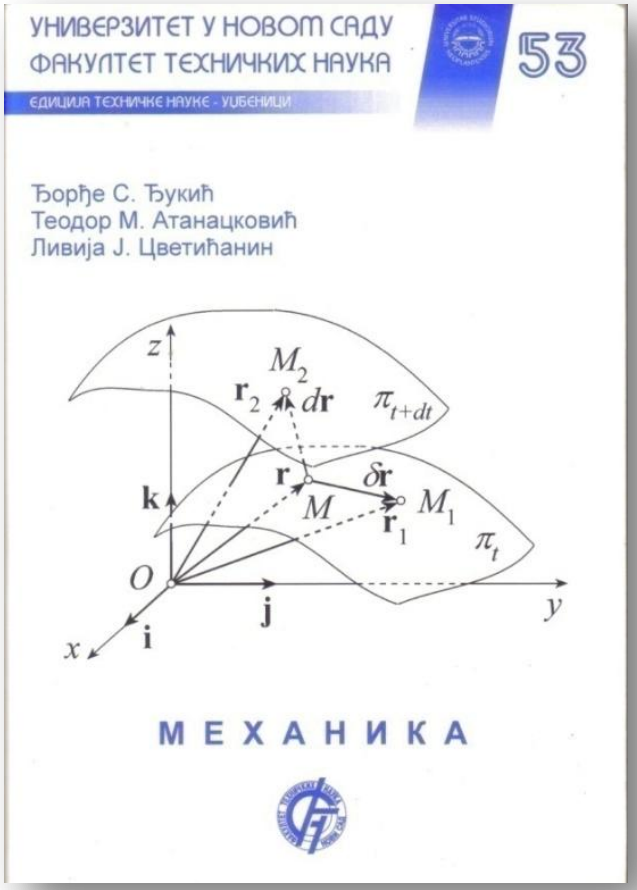
Mehanika 2 (Kinematika)
Predavanja 3

Miodrag Zuković

Novi Sad, 2023.

Literatura

- Đorđe S. Đukić, Teodor M. Atanacković, Livija J. Cvetićanin: Mehanika, Fakultet tehničkih nauka u Novom Sadu, Novi Sad, 2003.



Ispit

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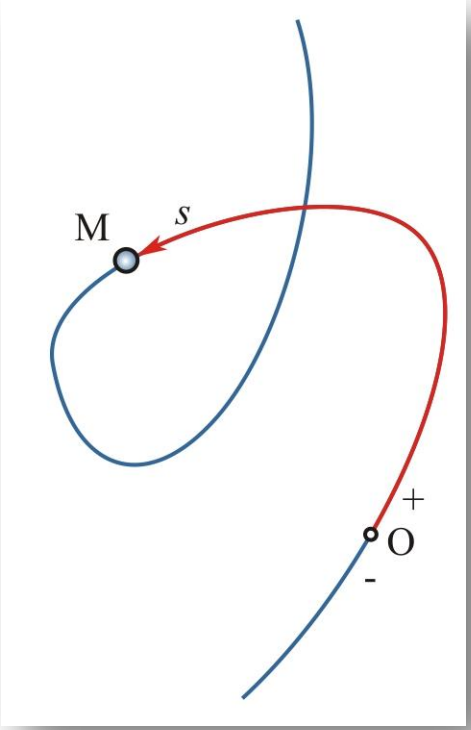
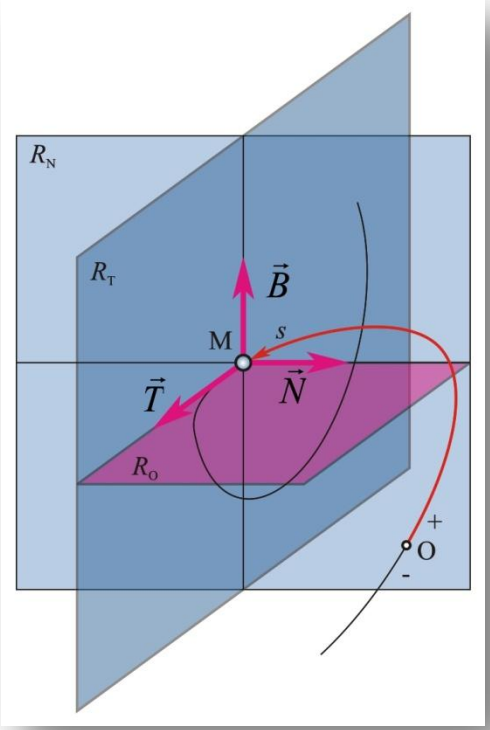
Šta ćemo naučiti?

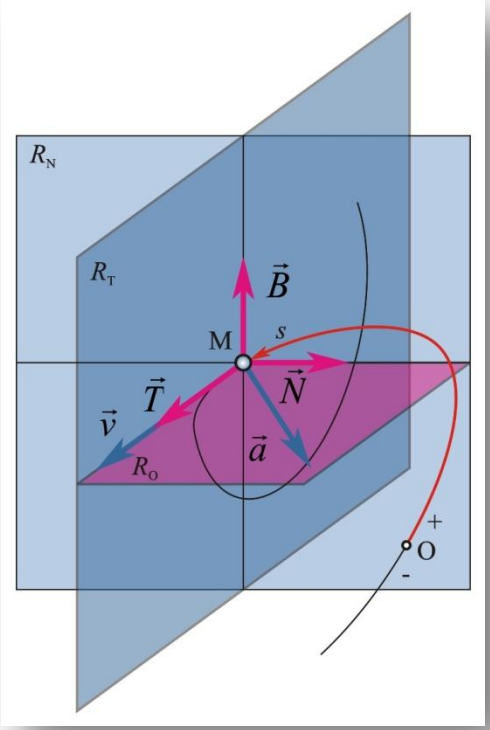
7. Kinematika tačke - prirodni koordinatni sistem
8. Kretanje tačke po kružnici

7. Kinematika tačke - prirodni koordinatni sistem

Prirodne koordinate







s – prirodna koordinata
$s(t)$ – zakon kretanja

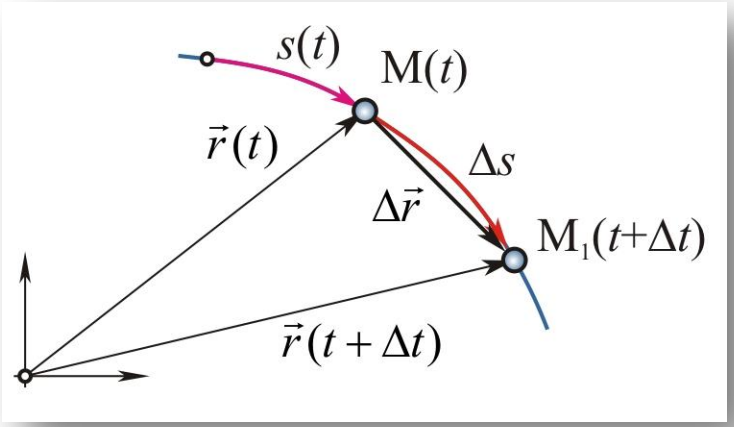
$$\vec{B} = \vec{T} \times \vec{N}$$

R_O – oskulatorna ravan
R_T – tangentna ravan
R_O – normalna ravan

Prirodni koordinatni sistem – brzina

$$\vec{v} = \dot{\vec{r}} = \frac{d\vec{r}}{dt}$$

$$\vec{v} = \frac{ds}{dt}\frac{d\vec{r}}{ds}$$



$$\frac{d\vec{r}}{ds} = \left|\frac{d\vec{r}}{ds}\right|\vec{T} = \frac{|d\vec{r}|}{ds}\vec{T}$$

$$\frac{|d\vec{r}|}{ds} = \lim_{\Delta s \to 0}\frac{|\Delta\vec{r}|}{\Delta s} = 1$$

$$\frac{d\vec{r}}{ds} = \vec{T}$$

$$\vec{v} = \frac{ds}{dt}\vec{T} = \dot{s}\,\vec{T} = v_T\vec{T}$$

$$v_T = \dot{s}$$

$$|\vec{v}| = v = |\dot{s}|$$

$$v^2 = \dot{s}^2$$

Prirodni koordinatni sistem – ubrzanje

$$\vec{v} = \dot{s}\,\vec{T}$$

$$\vec{a} = \dot{\vec{v}}$$

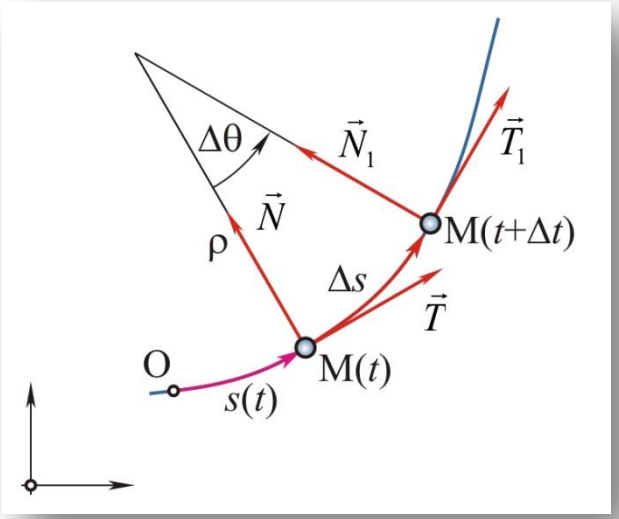
$$\vec{a} = \ddot{s}\,\vec{T} + \dot{s}\,\dot{\vec{T}}$$

$$\dot{\vec{T}} = \frac{d\vec{T}}{dt} = \frac{ds}{dt}\frac{d\theta}{ds}\frac{d\vec{T}}{d\theta}$$

$$\frac{ds}{dt} = \dot{s}$$

$$\frac{d\theta}{ds} = \chi = \frac{1}{R_k}$$

$$\frac{d\vec{T}}{d\theta} = ?$$



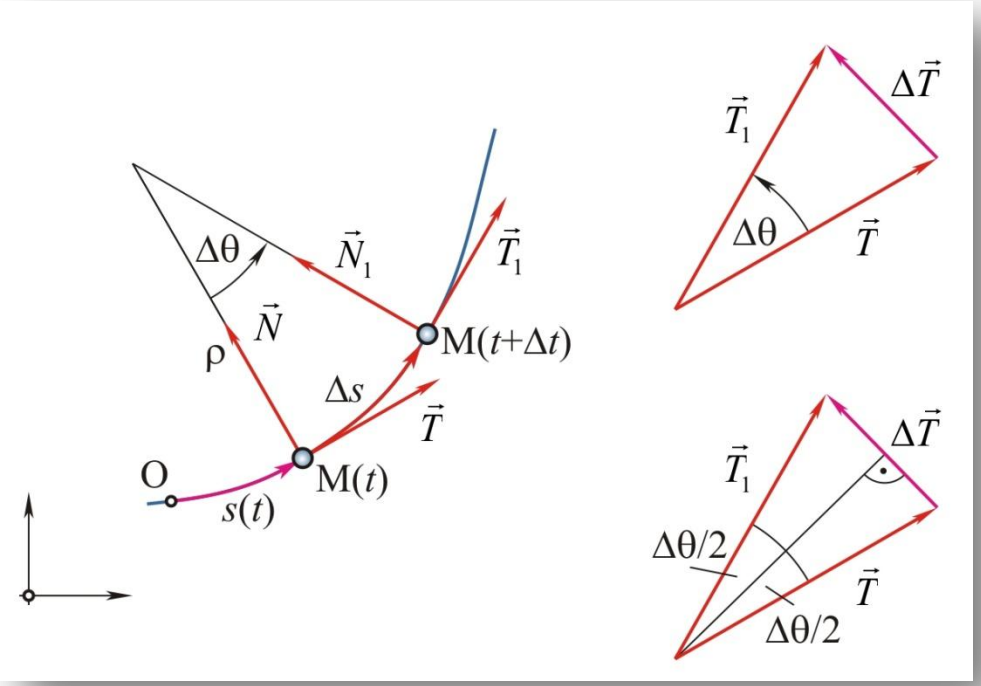
Prirodni koordinatni sistem – ubrzanje

$$\frac{d\vec{T}}{d\theta} = ?$$

$$\vec{T}\cdot\vec{T} = 1$$

$$\frac{d}{d\theta}(\vec{T}\cdot\vec{T}) = \frac{d\vec{T}}{d\theta}\cdot\vec{T} + \vec{T}\cdot\frac{d\vec{T}}{d\theta} = 0$$

$$\frac{d\vec{T}}{d\theta} \perp \vec{T} \rightarrow \frac{d\vec{T}}{d\theta} = \left|\frac{d\vec{T}}{d\theta}\right|\vec{N}$$



$$\left|\frac{d\vec{T}}{d\theta}\right| = \lim_{\Delta\theta\to 0}\frac{\left|\Delta\vec{T}\right|}{\Delta\theta} = \lim_{\Delta\theta\to 0}\frac{2\sin\frac{\Delta\theta}{2}}{\Delta\theta} = 1 \rightarrow \frac{d\vec{T}}{d\theta} = \vec{N}$$

Prirodni koordinatni sistem – ubrzanje

$$\vec{a} = \ddot{s}\,\vec{T} + \dot{s}\,\dot{\vec{T}}$$

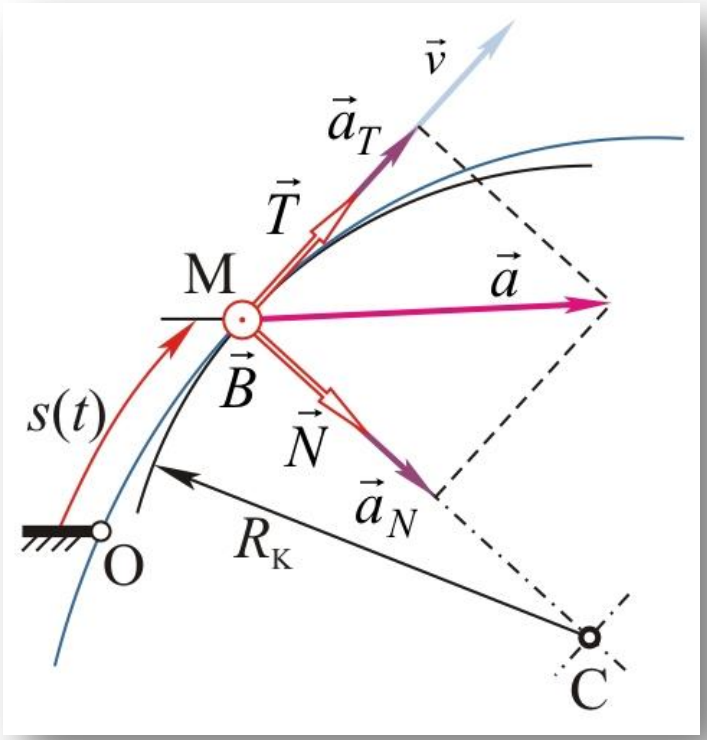
$$\dot{\vec{T}} = \frac{ds}{dt}\frac{d\theta}{ds}\frac{d\vec{T}}{d\theta} = \dot{s}\frac{1}{R_k}\vec{N}$$

$$\vec{a} = \ddot{s}\,\vec{T} + \frac{\dot{s}^2}{R_k}\vec{N}$$

$$\vec{a} = a_T\,\vec{T} + a_N\,\vec{N}$$

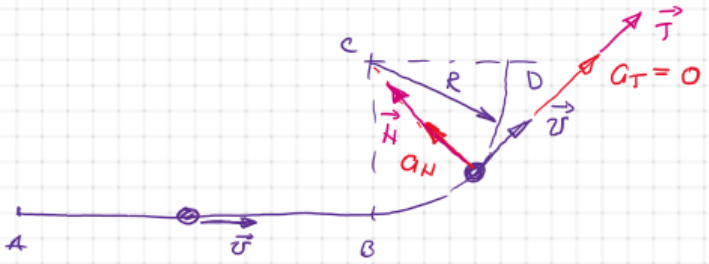
$$a_T = \ddot{s}$$

$$a_N = \frac{\dot{s}^2}{R_k} = \frac{v^2}{R_k}$$



$$|\vec{a}| = a = \sqrt{a_T^2 + a_N^2}$$

$v = V = const$



A-B, $\vec{v} = \overrightarrow{const}$

$\vec{a} = \dot{\vec{v}} = 0$

B-D, $v = V = const$

$\vec{v} \neq \overrightarrow{const}$

$\vec{a} = \dot{\vec{v}} \neq 0$

$\vec{a} = \vec{a}_T + \vec{a}_N$

$a_T = \ddot{s}$, $v = |\dot{s}|$, $v = const \rightarrow \dot{s} = const$

$\ddot{s} = 0$

$\boxed{a_T = 0}$

$a_N = \frac{\dot{s}^2}{R_k} = \frac{v^2}{R}$

$R_k = R = const$

$\vec{a} = \vec{a}_T$

$R_k = \infty$ $\quad a_N = 0$

- Poluprečnik krivine trajektorije određuje poluprečnik kružnice čiji se centar nalazi na pravcu normale na trajektoriju, ima zajedničku tangentu sa njom i najbolje aproksimira krivu u okolini posmatrane tačke

- Poluprečnik krivine kružnice jednak je samom poluprečniku kružnice, a poluprečnik krivine prave je beskonačno velik (krivina je jednaka nuli)

Poluprečnik krivine trajektorije

$$\vec{a} = a_T \vec{T} + a_N \vec{N}$$

$$\dot{s}^2 = v^2 \quad / \frac{d}{dt}$$

$$2\dot{s}\ddot{s} = 2v\dot{v}$$

$$a_T = \ddot{s}$$

$$a_N = \frac{\dot{s}^2}{R_k} = \frac{v^2}{R_k}$$

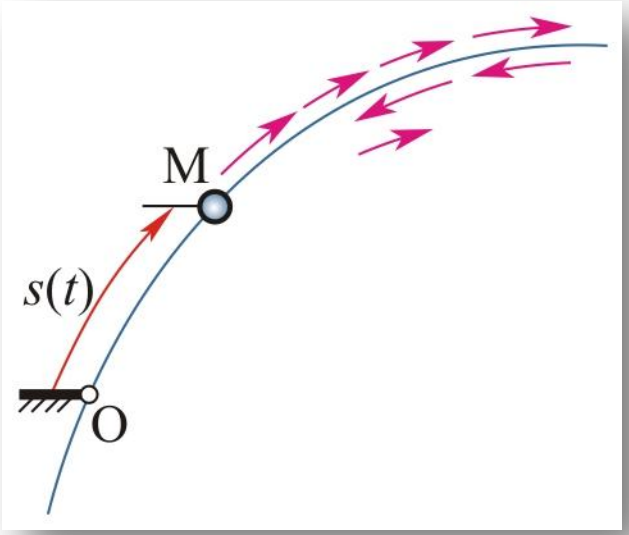
$$a_T^2 = \ddot{s}^2 = \frac{v^2}{\dot{s}^2}\dot{v}^2 = \dot{v}^2$$

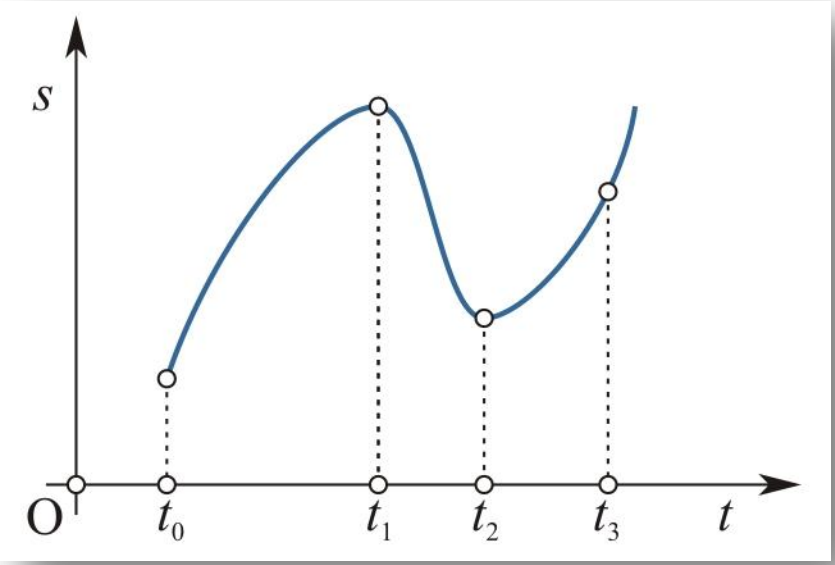
$$a = \sqrt{a_T^2 + a_N^2}$$

$$a_N = \sqrt{a^2 - a_T^2}$$

$$R_k = \frac{v^2}{a_N}$$

Pređeni put





$$s(t) = \pm \int_{t_0}^{t} v(t)dt$$

$$P[t_0, t_3] = P[t_0, t_1] + P[t_1, t_2] + P[t_2, t_3]$$

$$P[t_0, t_3] = |s(t_1) - s(t_0)| + |s(t_2) - s(t_1)| + |s(t_3) - s(t_2)|$$

Primer

Kretanje tačke je opisano parametarskim jednačinama

$$x(t)=\frac{t^2}{2}-t \qquad y(t)=\frac{t}{2}$$

a) Odrediti trajektoriju tačke,
b) odrediti trenutak $t^*>0$ u kome će se tačka naći na osi y,
c) odrediti brzinu i ubrzanje tačke u proizvoljnom trenutku vremena t,
d) odrediti brzinu i ubrzanje tačke, i njihove intenzitete, u trenutku t^*,
e) odrediti poluprečnik krivine trajektorije u trenutku t^*.

$x(t)=\frac{t^2}{2}-t \quad y(t)=\frac{t}{2}$

ЛП

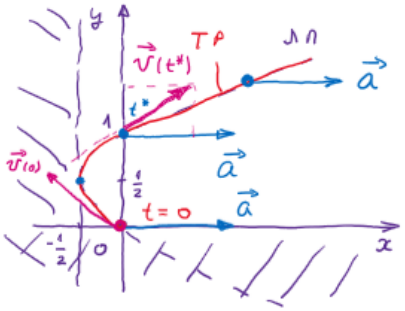
$x(t)$
$y(t)$ } ел. $\xrightarrow{t}$ $y(x)$

$t=2y \rightarrow x=\frac{(2y)^2}{2}-2y$

$\boxed{x=2y^2-2y}$

ПАРАБОЛА

$x=2y(y-1)$



$t=0$
УСЛОВНО

$t^*=2$
УБРАЗНО

Ok $\quad t \geq 0$

(2) $y=\frac{t}{2} \rightarrow y \geq 0$

(1) $x=t\left(\frac{t}{2}-1\right) \rightarrow -\frac{1}{2} \leq x$

$t^*=?$
$t^*>0$ $\rightarrow$ на y оси $\rightarrow$ $x(t^*)=0$

$x(t^*)=t^*\left(\frac{t^*}{2}-1\right)=0$

$t^*=0$

$\boxed{t^*=2}$

$t^*=2 \rightarrow x(t^*)=x(2)=\frac{2^2}{2}-2=0$

$y(t^*)=y(2)=\frac{2}{2}=1$

$x(t)=\frac{t^2}{2}-t \quad y(t)=\frac{t}{2}$

$\vec{v}=\dot{\vec{r}}=\dot{x}\vec{i}+\dot{y}\vec{j}$

$\vec{a}=\dot{\vec{v}}=\ddot{\vec{r}}=\ddot{x}\vec{i}+\ddot{y}\vec{j}$

$\dot{x}(t)=t-1 \; ; \; \dot{y}(t)=\frac{1}{2}=const$

$v(t)=\sqrt{\dot{x}^2(t)+\dot{y}^2(t)}=\sqrt{(t-1)^2+\left(\frac{1}{2}\right)^2}$

$\ddot{x}(t)=1=const \; ; \; \ddot{y}(t)=0=const$

$\vec{a}_{(t)}=1\cdot\vec{i}+0\vec{j}=1\cdot\vec{i}=\overrightarrow{const}$

$a(t)=\sqrt{\ddot{x}^2(t)+\ddot{y}^2(t)}=1=const$

$t^*=2 \qquad x(2)=0 \; , \; y(2)=1$

$\dot{x}(t^*)=\dot{x}(2)=2-1=1$

$\dot{y}(t^*)=\dot{y}(2)=\frac{1}{2}$

$v(t^*)=v(2)=\sqrt{1^2+\left(\frac{1}{2}\right)^2}=\frac{\sqrt{5}}{2}$

$\ddot{x}(2)=1 \; , \; \ddot{y}(2)=0$

$a(2)=1$

$$x(t)=\frac{t^2}{2}-t \qquad y(t)=\frac{t}{2}$$

$t^* = 2$

$v(2) = \sqrt{(2-1)^2 + \frac{1}{4}} = \frac{\sqrt{5}}{2}$

$R_k(t^*) = R_k(2) = ?$

$a(2) = 1$

$\dot{x}(t) = t - 1 \quad , \quad \dot{y}(t) = \frac{1}{2}$

$a_T^2(2) = \dot{v}^2(2) = \left(\frac{2}{\sqrt{5}}\right)^2 = \frac{4}{5}$

$v_{(t)} = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{(t-1)^2 + \left(\frac{1}{2}\right)^2} \quad / \frac{d}{dt} \rightarrow \quad \dot{v}(t) = \frac{2(t-1)\cdot 1}{2\sqrt{(t-1)^2 + \frac{1}{4}}}$

$\ddot{x}(t) = 1 \quad , \quad \ddot{y}(t) = 0$

$\dot{v}(2) = \frac{2-1}{\sqrt{(2-1)^2 + \frac{1}{4}}} = \frac{1}{\frac{\sqrt{5}}{2}} = \frac{2}{\sqrt{5}}$

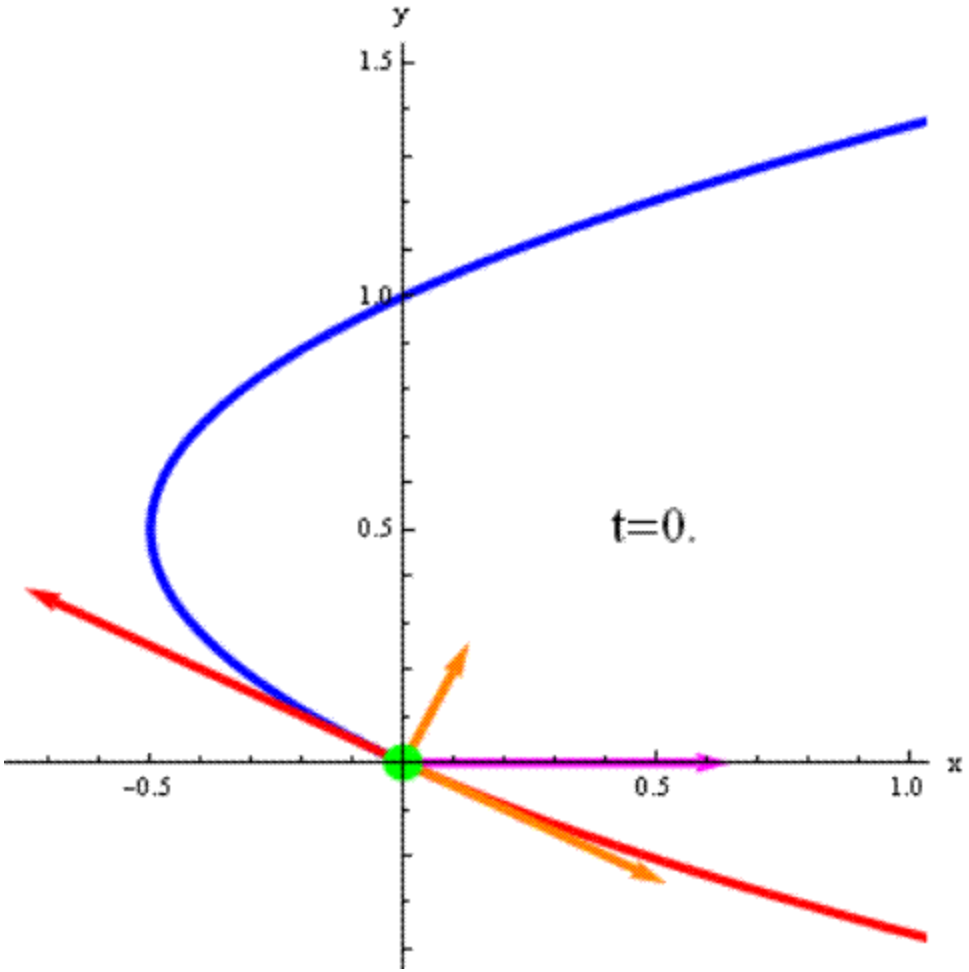
$a(t) = \sqrt{\ddot{x}^2 + \ddot{y}^2} = 1 = const$

$a_N(2) = \sqrt{a^2(2) - a_T^2(2)} = \sqrt{1^2 - \frac{4}{5}}$

$= \sqrt{1 - \frac{4}{5}} = \sqrt{\frac{1}{5}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$

$R_k(2) = \frac{v^2(2)}{a_N(2)} = \frac{\left(\frac{\sqrt{5}}{2}\right)^2}{\frac{1}{\sqrt{5}}} = \frac{\frac{5}{4}}{\frac{1}{\sqrt{5}}} = \frac{5\sqrt{5}}{4} \quad [m]$

$$R_k(t^*) = \frac{v^2(t^*)}{a_n(t^*)} = \frac{5\sqrt{5}}{4}\, m.$$


y
1.5
1.0
0.5
t=0.
-0.5
0.5
1.0
x

Primer

Kretanje tačke je opisano parametarskim jednačinama

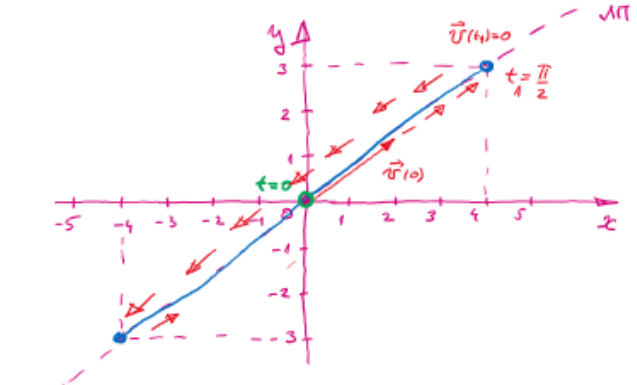
$$x(t) = 4\sin t\,, y(t) = 3\sin t$$

a) Odrediti i nacrtati trajektoriju kretanja tačke,
b) odrediti brzinu i ubrzanje tačke u proizvoljnom trenutku vremena t,
c) odrediti položaj, brzinu i ubrzanje tačke u trenucima $t_0 = 0, t_1 = \pi/2$ i $t_2 = \pi$,
d) odrediti pređeni put tačke do trenutka t_2.

Kretanje tačke je opisano parametarskim jednačinama

$$x(t) = 4\sin t\,,\; y(t) = 3\sin t$$

a) Odrediti i nacrtati trajektoriju kretanja tačke,
b) odrediti brzinu i ubrzanje tačke u proizvoljnom trenutku vremena t,
c) odrediti položaj, brzinu i ubrzanje tačke u trenucima $t_0 = 0, t_1 = \pi/2$ i $t_2 = \pi$,



a) (1) $x = 4\sin t \longrightarrow \sin t = \frac{x}{4}$

(2) $y = 3\sin t$

$y = 3 \cdot \frac{x}{4}$

$\boxed{y = \frac{3}{4}x}$ ΔΠ

ПРАВА

Ok $t \geq 0$

(1) → $-4 \leq x \leq 4$

(2) → $-3 \leq y \leq 3$

b) (1) $x(t) = 4\sin t$

(2) $y(t) = 3\sin t$

$\dot{x}(t) = 4\cos t$

$\dot{y}(t) = 3\cos t$

$v(t) = \sqrt{4^2\cos^2 t + 3^2\cos^2 t}$

$v(t) = \sqrt{25\cos^2 t}$

$v(t) = |v(t)| = 5\sqrt{\cos^2 t}$

$\ddot{x}(t) = -4\sin t$

$\ddot{y}(t) = -3\sin t$

$a(t) = \sqrt{4^2\sin^2 t + 3^2\sin^2 t}$

$a(t) = 5\sqrt{\sin^2 t}$

Одредити смер $\vec{v}$ кретања у почетном тренутку.

$t = 0$

$x(0) = 4\sin 0 = 0$

$y(0) = 4\sin 0 = 0$

$\dot{x}(0) = 4\cos 0 = 4$

$\dot{y}(0) = 3$

$v(0) = 5\sqrt{\cos^2 0} = 5$

$\ddot{x}(0) = 0$

$\ddot{y}(0) = 0$

$a(0) = 0$

Kretanje tačke je opisano parametarskim jednačinama

$$x(t) = 4\sin t\,,\; y(t) = 3\sin t$$

a) Odrediti i nacrtati trajektoriju kretanja tačke,
b) odrediti brzinu i ubrzanje tačke u proizvoljnom trenutku vremena t,
c) odrediti položaj, brzinu i ubrzanje tačke u trenucima $t_0 = 0, t_1 = \pi/2$ i $t_2 = \pi$,
d) odrediti pređeni put tačke do trenutka t_2.

$P[t_0=0,\ t_1=\pi] = ?$

$\dot{x}(t) = 4\cos t$

$\dot{y}(t) = 3\cos t$

$v(t) = \sqrt{4^2\cos^2 t + 3^2\cos^2 t} = 5\sqrt{\cos^2 t}$

$s(t_0=0) = 0$

$s(t) = \pm\int_{t_0=0}^{t} v(t)\,dt = +\int_{t_0=0}^{t} 5\cos t\,dt$

$s(t) = 5\sin t\Big|_0^t = 5(\sin t - \sin 0)$

$s(t) = 5\sin t \rightarrow \dot{s}(t) = 5\cos t \rightarrow \dot{s}(t^*) = 5\cos t^* = 0$

$\cos t^* = 0$

$t^* = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$

$P[t_0=0,\ t_2=\pi] = P[0,\tfrac{\pi}{2}] + P[\tfrac{\pi}{2},\pi]$

$= |s(\tfrac{\pi}{2}) - s(0)| + |s(\pi) - s(\tfrac{\pi}{2})|$

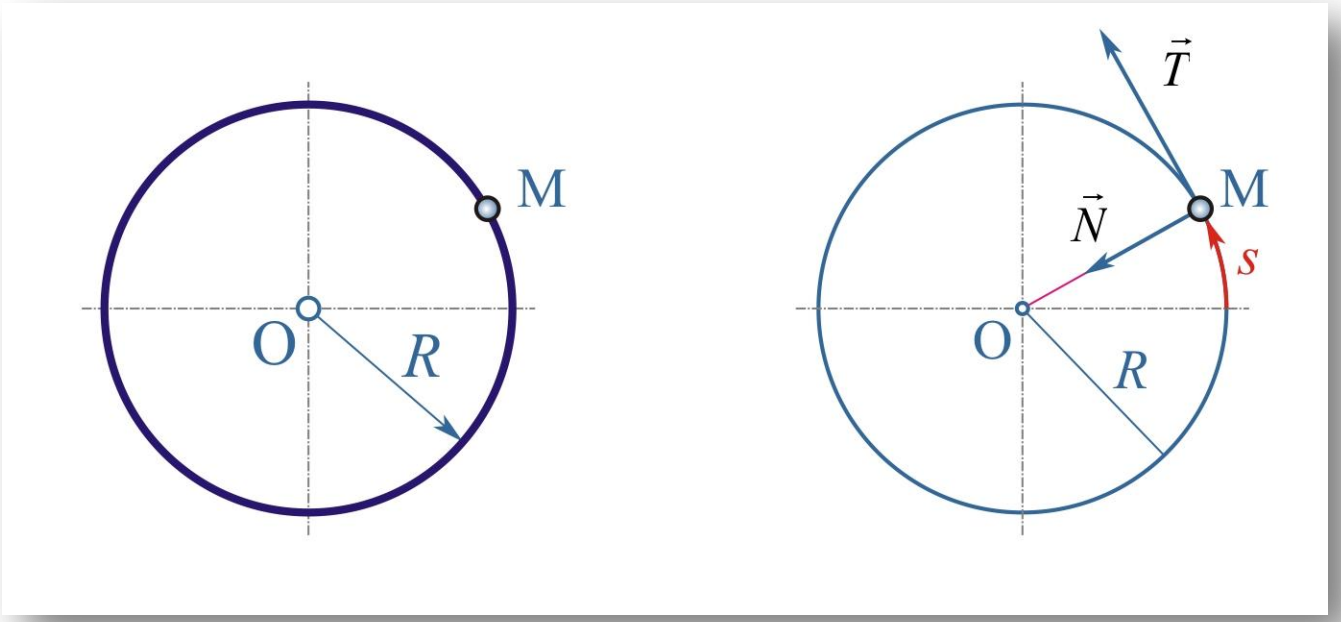
$P[0,\pi] = |5-0| + |0-5| = 5 + 5 = 10\ [m]$

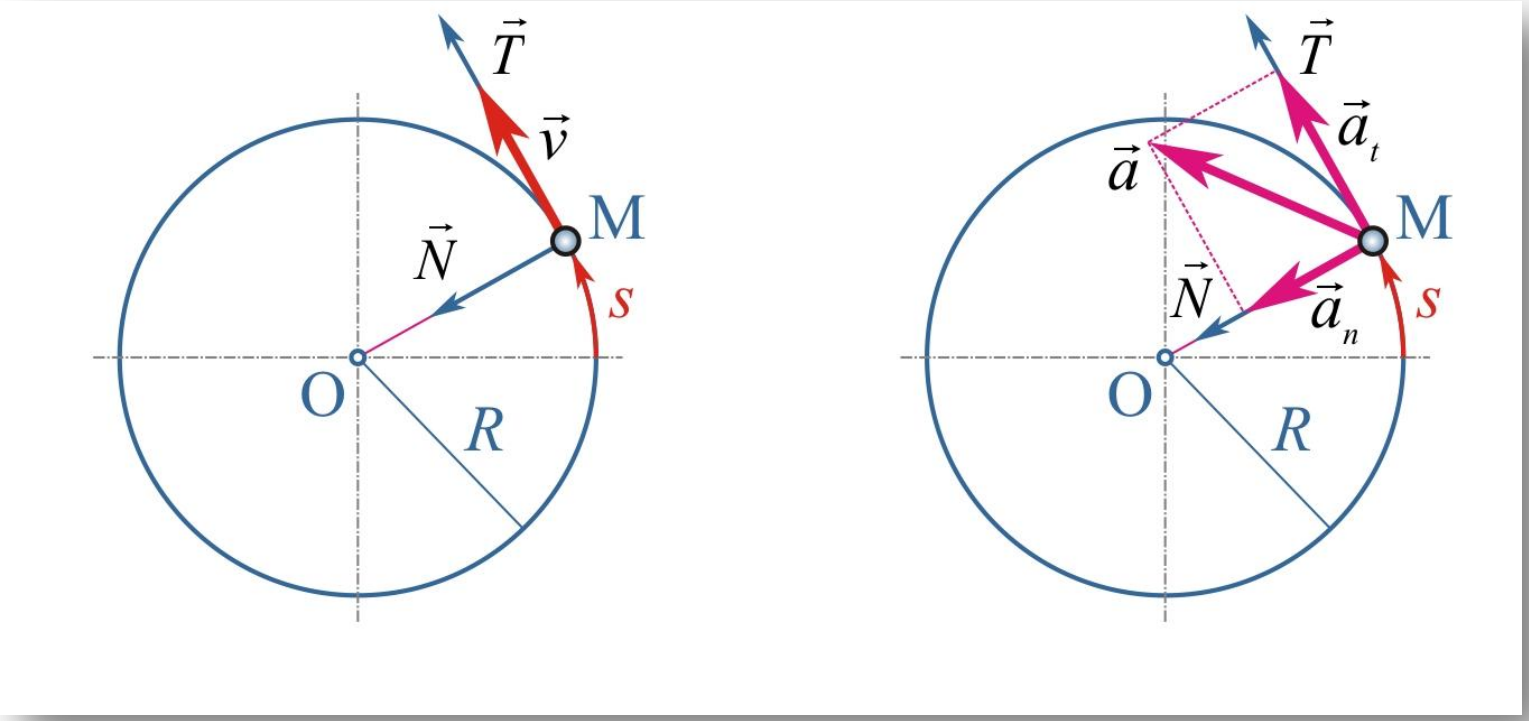
$s(0) = 5\sin 0 = 0$

$s(\tfrac{\pi}{2}) = 5\sin\tfrac{\pi}{2} = 5$

$s(\pi) = 5\sin\pi = 0$

8. Kretanje tačke po kružnici


M
O
R
$\vec{T}$
$\vec{N}$
M
s
O
R



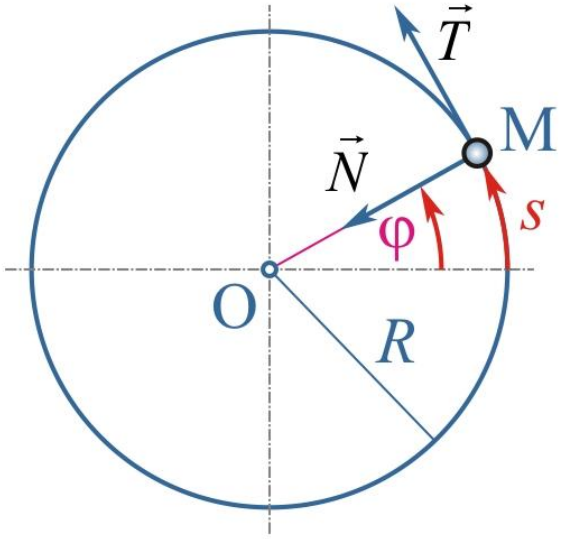
$$R_k = R = const$$

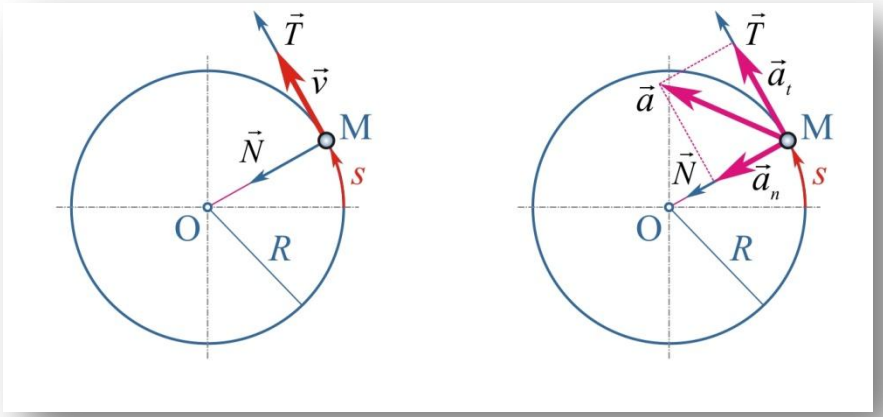
$$\vec{v} = \dot{s}\,\vec{T}$$

$$\vec{a} = \ddot{s}\,\vec{T} + \frac{\dot{s}^2}{R}\,\vec{N}$$

$$a_T = \ddot{s}$$

$$a_N = \frac{\dot{s}^2}{R} = \frac{v^2}{R}$$





$$s = R\varphi$$
$$\dot{s} = R\dot{\varphi}$$
$$\ddot{s} = R\ddot{\varphi}$$

$$\vec{v} = \dot{s}\,\vec{T} = R\dot{\varphi}\,\vec{T}$$

$$\vec{a} = \ddot{s}\,\vec{T} + \frac{\dot{s}^2}{R}\,\vec{N}$$
$$\vec{a} = R\ddot{\varphi}\,\vec{T} + R\dot{\varphi}^2\,\vec{N}$$

$$a_T = \ddot{s} = R\ddot{\varphi}$$
$$a_N = \frac{\dot{s}^2}{R} = \frac{v^2}{R} = R\dot{\varphi}^2$$

Primer

Пример 4.4 *Материјална тачка врши кретање по кругу полупречника* R *сагласно следећим параметарсим једначинама:*

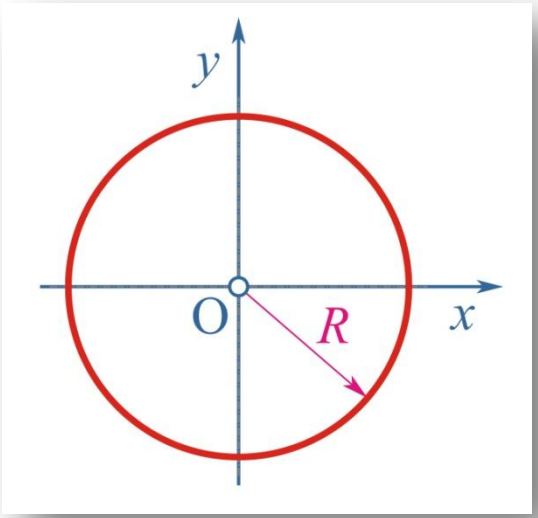
$$x(t) = R\cos\varphi(t); \quad y(t) = R\sin\varphi(t),$$

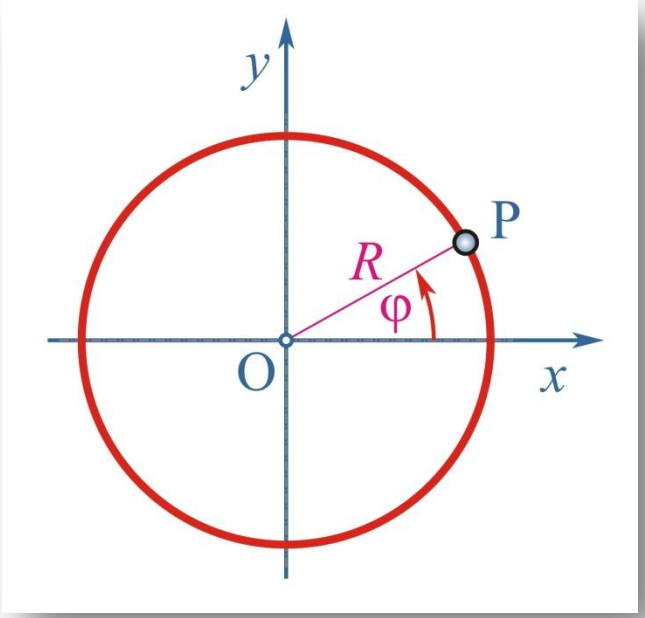
где је $\varphi(t)$ *два пута диференцијабилна функција времена* t. *Одредити векторе брзине и убрзања у призвољном тренутку времена и одредити њихове пројекције на правац тангенте и правац нормале на круг.*

Trajektorija

$$\left.\begin{aligned} x &= R\cos\varphi \\ y &= R\sin\varphi \end{aligned}\right\} \rightarrow \left.\begin{aligned} x^2 &= R^2\cos^2\varphi \\ y^2 &= R^2\sin^2\varphi \end{aligned}\right\} \rightarrow x^2 + y^2 = R^2\left(\cos^2\varphi + \sin^2\varphi\right)$$

$$x^2 + y^2 = R^2$$



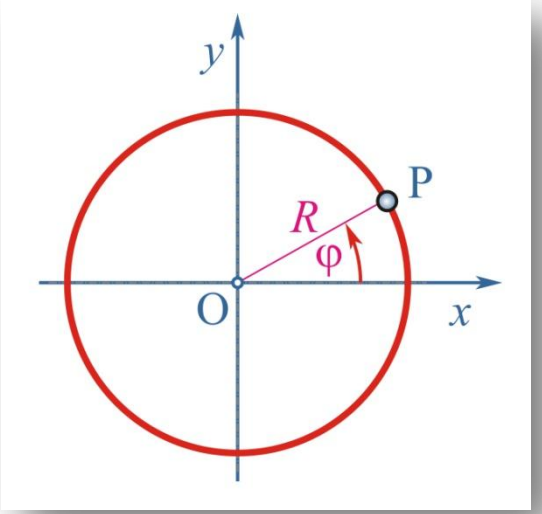


Brzina i ubrzanje

$$\left.\begin{aligned} x(t) &= R\cos\varphi(t) \\ y(t) &= R\sin\varphi(t) \end{aligned}\right\} \rightarrow \begin{aligned} \dot{x}(t) &= -R\sin\varphi(t)\cdot\dot{\varphi}(t) \\ \dot{y}(t) &= R\cos\varphi(t)\cdot\dot{\varphi}(t) \\ \ddot{x}(t) &= -R\cos\varphi(t)\cdot\dot{\varphi}^2(t) - R\sin\varphi(t)\cdot\ddot{\varphi}(t) \\ \ddot{y}(t) &= -R\sin\varphi(t)\cdot\dot{\varphi}^2(t) + R\cos\varphi(t)\cdot\ddot{\varphi}(t) \end{aligned}$$

$$\vec{v}(t) = \dot{x}(t)\vec{i} + \dot{y}(t)\vec{j}$$

$$\vec{a}(t) = \ddot{x}(t)\vec{i} + \ddot{y}(t)\vec{j}$$



Brzina

$$\vec{v}(t) = \left(-R\sin\varphi(t)\cdot\dot{\varphi}(t)\right)\vec{i} + \left(R\cos\varphi(t)\cdot\dot{\varphi}(t)\right)\vec{j}$$

$$\vec{v}(t) = \left(-\sin\varphi(t)\vec{i} + \cos\varphi(t)\,\vec{j}\right)R\dot{\varphi}(t)$$

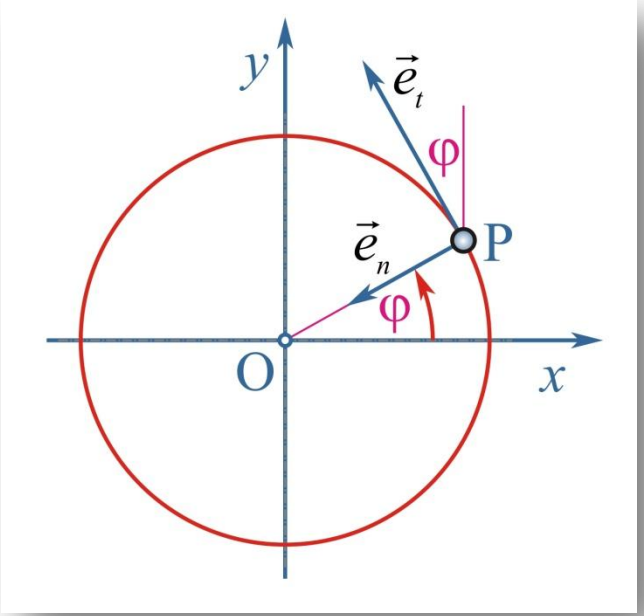
$$\vec{e}_t(t) = -\sin\varphi(t)\vec{i} + \cos\varphi(t)\,\vec{j}$$

$$\vec{e}_n(t) = -\cos\varphi(t)\vec{i} - \sin\varphi(t)\,\vec{j}$$

$$\vec{v}(t) = R\dot{\varphi}(t)\cdot\vec{e}_t(t)$$

$$\vec{v}(t) = v_t(t)\cdot\vec{e}_t(t)$$

$$v_t(t) = R\dot{\varphi}(t)$$



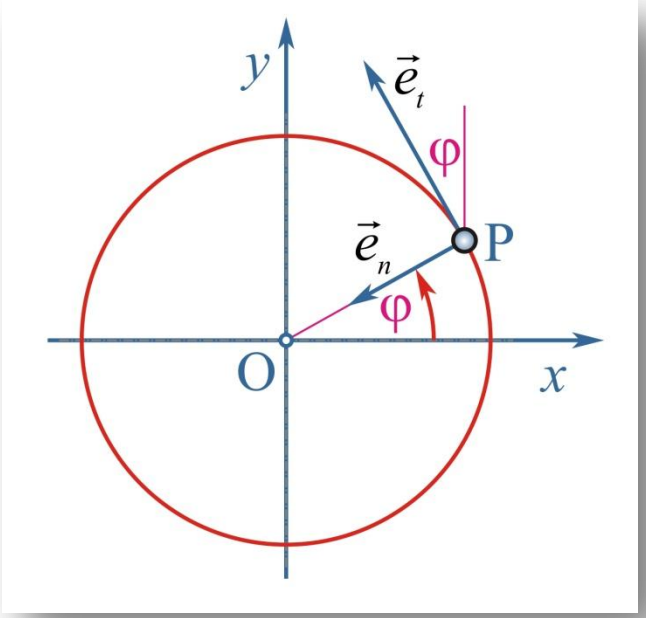
Ubrzanje

$$\vec{a}(t)=\left(-R\cos\varphi(t)\cdot\dot{\varphi}^2(t)-R\sin\varphi(t)\cdot\ddot{\varphi}(t)\right)\vec{i}+\left(-R\sin\varphi(t)\cdot\dot{\varphi}^2(t)+R\cos\varphi(t)\cdot\ddot{\varphi}(t)\right)\vec{j}$$

$$\vec{a}(t)=\left(-R\sin\varphi(t)\vec{i}+\cos\varphi(t)\,\vec{j}\right)R\ddot{\varphi}(t)+\left(-\cos\varphi(t)\vec{i}-\sin\varphi(t)\,\vec{j}\right)R\dot{\varphi}^2(t)$$

$$\vec{e}_t(t)=-\sin\varphi(t)\vec{i}+\cos\varphi(t)\,\vec{j}$$

$$\vec{e}_n(t)=-\cos\varphi(t)\vec{i}-\sin\varphi(t)\,\vec{j}$$



$$\vec{a}(t)=\left(R\ddot{\varphi}(t)\right)\cdot\vec{e}_t(t)+\left(R\dot{\varphi}^2(t)\right)\cdot\vec{e}_n(t)$$

$$\vec{a}(t)=a_t(t)\cdot\vec{e}_t(t)+a_n(t)\cdot\vec{e}_n(t)$$

$$a_t(t)=R\ddot{\varphi}(t)$$

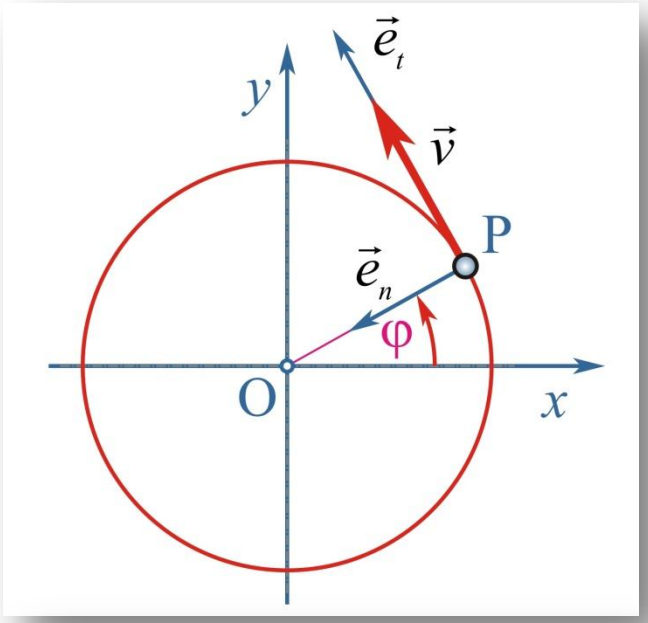
$$a_n(t)=R\dot{\varphi}^2(t)$$

Brzina i ubrzanje

$$\vec{v}(t) = R\dot{\varphi}(t) \cdot \vec{e}_t(t)$$

$$\vec{v}(t) = v_t(t) \cdot \vec{e}_t(t)$$

$$v_t(t) = R\dot{\varphi}(t)$$

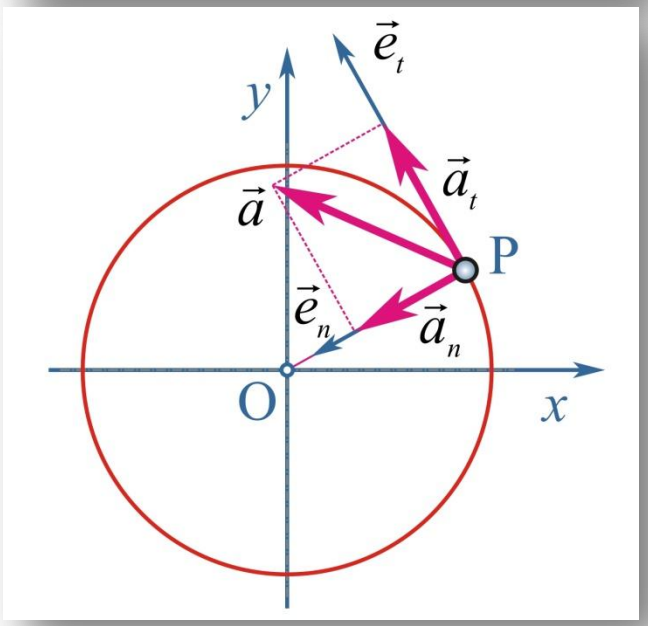


$$\vec{a}(t) = \left(R\ddot{\varphi}(t)\right) \cdot \vec{e}_t(t) + \left(R\dot{\varphi}^2(t)\right) \cdot \vec{e}_n(t)$$

$$\vec{a}(t) = a_t(t) \cdot \vec{e}_t(t) + a_n(t) \cdot \vec{e}_n(t)$$

$$a_t(t) = R\ddot{\varphi}(t)$$

$$a_n(t) = R\dot{\varphi}^2(t)$$



Šta smo naučili?

7. Kinematika tačke - prirodni koordinatni sistem
8. Kretanje tačke po kružnici

Mehanika 2 (Kinematika)
Predavanja 3

Miodrag Zuković

Novi Sad, 2023.